

Math 1B: Calculus (Fall 2014)

Lecture 15: Sequences

A sequence is an infinite list of numbers, plain and simple!

We usually write a sequence in the form

$$\{a_1, a_2, a_3, \dots\}$$

↑ ↑ ↑
1st term 2nd term 3rd term

Alternate Notation: $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

We can define a sequence in many different ways:

1) By a formula or function, e.g.

$$\left\{ \frac{n!}{n+1} \right\}_{n=1}^{\infty} = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{6}{4}, \frac{24}{5}, \dots \right\}$$

$$\left\{ \sin\left(\frac{n\pi}{2}\right) \right\}_{n=1}^{\infty} = \{1, 0, -1, 0, 1, 0, \dots\}$$

2) By a recursive formula, e.g. the Fibonacci sequence

$$\{1, 1, 2, 3, 5, 8, 13, \dots\}, \text{ where } a_{n+2} = a_{n+1} + a_n$$

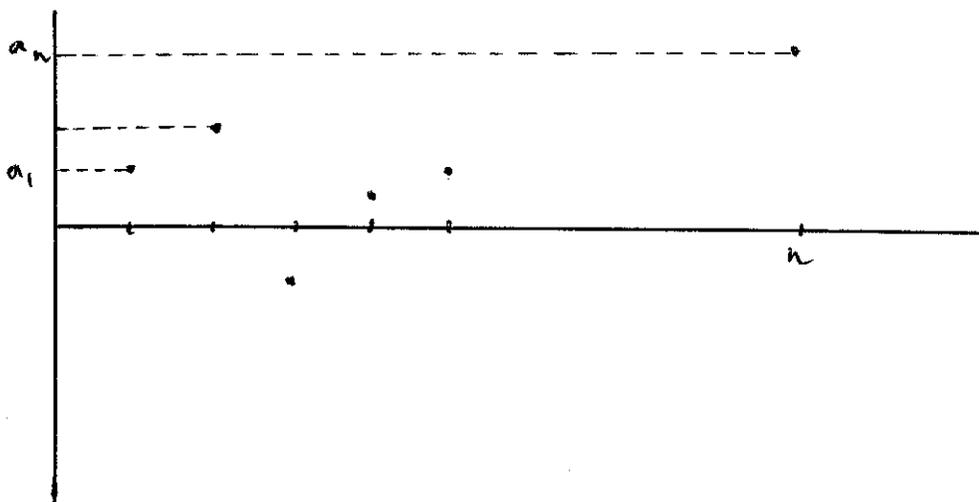
for all $n \geq 1$.

Good way to think of a sequence:

A sequence is a function where the only inputs are

$$\{1, 2, 3, 4, 5, \dots\}.$$

Like a function, it's useful to visualise a sequence as a collection of points in the x - y plane:



We use many of the same words to describe properties of sequences, as we use for functions.

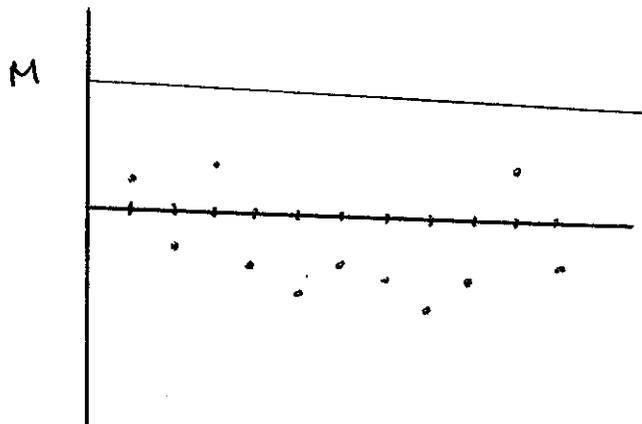
Definition

A sequence $\{a_n\}$ is bounded above if there exists a number M , such that $a_n < M$, for all $n \geq 1$.

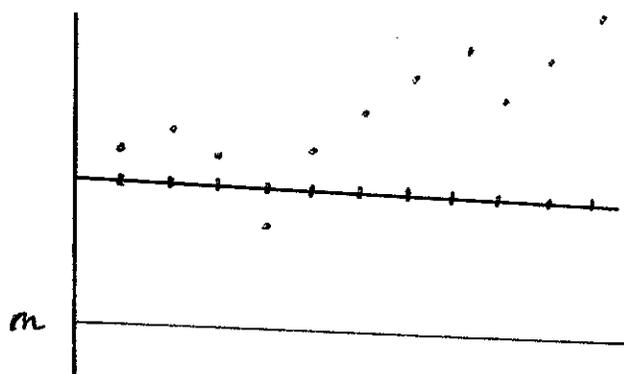
A sequence $\{a_n\}$ is bounded below if there exists a number m , such that $a_n > m$, for all $n \geq 1$.

A sequence $\{a_n\}$ is bounded if it is both bounded above and bounded below.

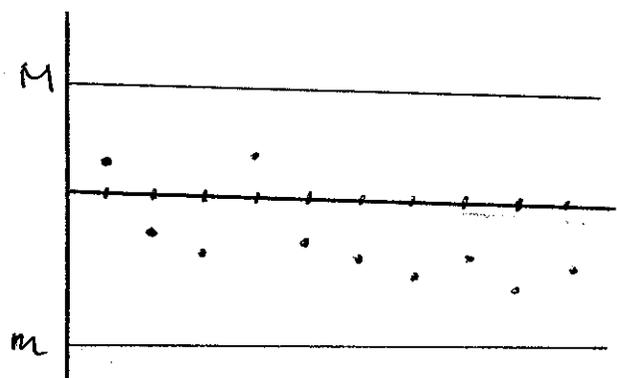
Each of these have a nice interpretation in terms of the graphs.



$\{a_n\}$ bounded above by M
means the sequence never
goes above the line $Y = M$.



$\{a_n\}$ bounded below by m
means the sequence never
goes below the line
 $Y = m$



$\{a_n\}$ bounded means the
sequence always stays in
a horizontal strip for some
 M and m .

A sequence can of course be bounded below, but not
above. For example $\{n\}_{n=1}^{\infty}$ is bounded below by
 -1 , but is not bounded above.

Question: Is every sequence bounded below or above?

Hint: Think about $\left\{n \sin\left(\frac{\pi n}{2}\right)\right\}_{n=1}^{\infty}$.

Definition A sequence $\{a_n\}_{n=1}^{\infty}$ is increasing, if

$$a_n < a_{n+1} \quad \text{for all } n \geq 1$$

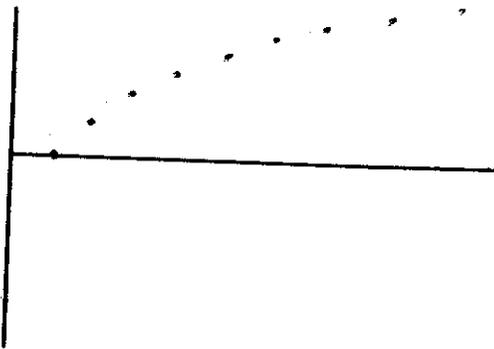
That is $a_1 < a_2 < a_3 \dots$. It is decreasing

if $a_{n+1} < a_n$ for all $n \geq 1$. That is

$$a_1 > a_2 > a_3 \dots$$

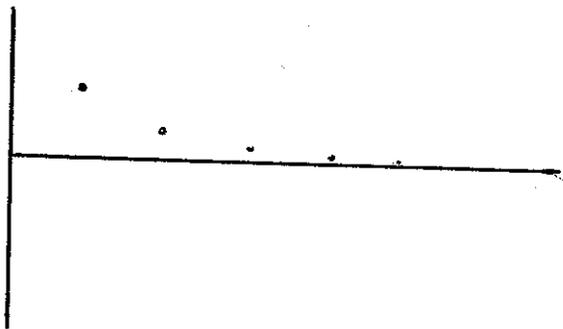
A sequence is monotonic if it is either increasing or decreasing.

Examples $\{\ln(n)\}_{n=1}^{\infty}$ is increasing



← Note how each point increases at each step.

$\{\frac{1}{n}\}_{n=1}^{\infty}$ is decreasing



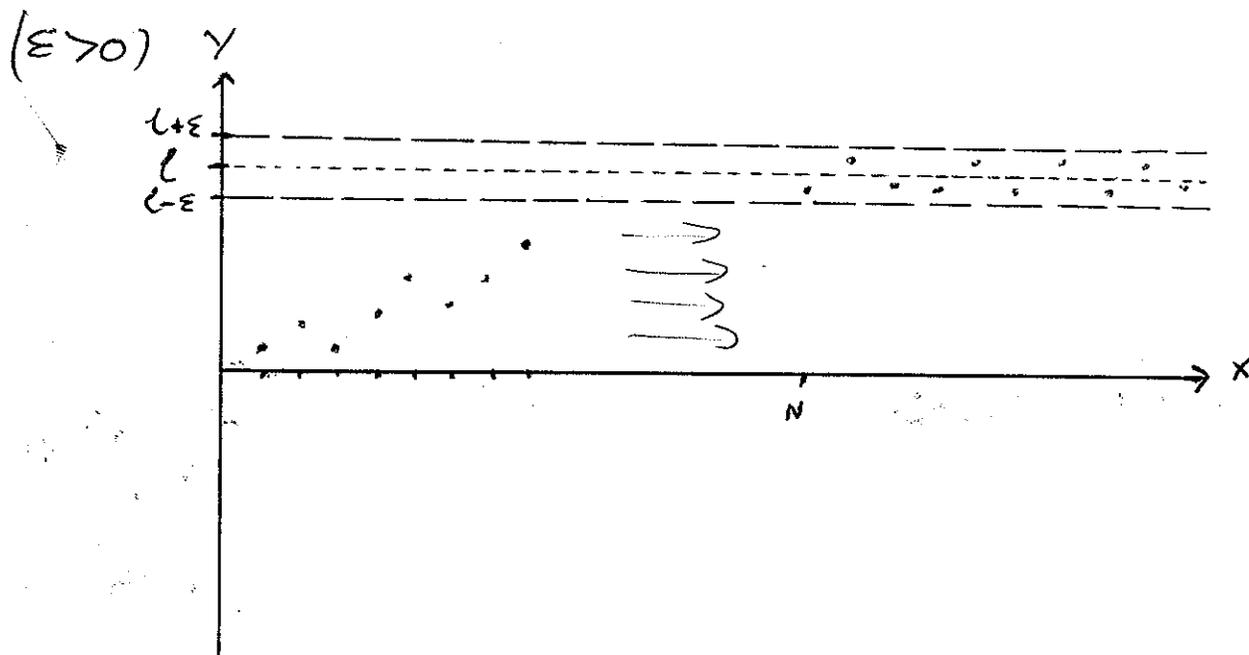
← Note how each point decreases at each step.

Question: Is the sequence $\{\frac{n}{n^2+1}\}_{n=1}^{\infty}$ monotone?

Limits of sequences

Let $\{a_1, a_2, a_3, \dots\}$ be a sequence. Intuitively we think of a limit of this sequence as being a number l that the sequence gets arbitrarily close to as $n \rightarrow \infty$.

Let's make this precise with a helpful picture



In words the above ~~statement~~ means the following: Given any narrow strip around the line $y = l$ (given by some $\epsilon > 0$) there is a number N such that for every $n > N$, a_n is contained in the strip.

Really precise statement: Given $\epsilon > 0$, there

exists N , a number, such that $l - \epsilon < a_n < l + \epsilon$ for all $n > N$.

If this is the case we say $\{a_n\}$ is convergent and that $\lim_{n \rightarrow \infty} \{a_n\} = l$.

example 1) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ is convergent and $\lim_{n \rightarrow \infty} \left\{\frac{n}{n+1}\right\} = 1$

2) $\pi = \lim_{n \rightarrow \infty} \{3, 3.1, 3.14, 3.141, 3.1415, \dots\}$,

If a sequence does not converge we say it diverges.

Sequences can diverge in different ways. For example

we say $\{a_n\}$ diverges to ∞ if the following occurs:

Given $M > 0$, there exists $N > 0$, such that

$a_n > M$ for all $n > N$.

Nice exercise: Draw a picture of this similar to the one on the previous page.

We have a similar definition for diverging to $-\infty$.

Question: Does $\left\{\sin\left(\frac{n\pi}{2}\right)\right\}_{n=1}^{\infty}$ converge or diverge. If it diverges does it diverge to either ∞ or $-\infty$?

On page 653 are a selection of properties of limits after adding and scaling sequences. Look at it.

Remember that any function $f(x)$ gives rise to a sequence $\{f(n)\}_{n=1}^{\infty}$.

Fact: If $\lim_{x \rightarrow \infty} f(x)$ exists, then $\lim_{n \rightarrow \infty} \{f(n)\} = \lim_{x \rightarrow \infty} f(x)$.

This is because the definition of a limit of a function as $x \rightarrow \infty$ is almost identical to the definition of the limit of a sequence.

Hence we can sometimes use calculus to determine limits of sequences. See Example 6 on page 644.

Squeeze Theorem If $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences such that

1) $a_n \leq b_n \leq c_n$ for all $n \geq 1$

2) $\{a_n\}, \{c_n\}$ convergent with $\lim_{n \rightarrow \infty} \{a_n\} = \lim_{n \rightarrow \infty} \{c_n\}$

Then $\{b_n\}$ convergent and

$$\lim_{n \rightarrow \infty} \{a_n\} = \lim_{n \rightarrow \infty} \{b_n\} = \lim_{n \rightarrow \infty} \{c_n\}$$

Monotonic Sequence Theorem (Very Important)

Any monotonic and bounded sequence is convergent.

Read example 14 to see this in action.