

Lecture 10: Area of a Surface of Revolution.

A surface of revolution is a surface obtained by taking a graph for a function $y=f(x)$ and rotating it about the x -axis in three dimensions.

Examples $y=x$ $0 \leq x \leq 1$ gives a cone.

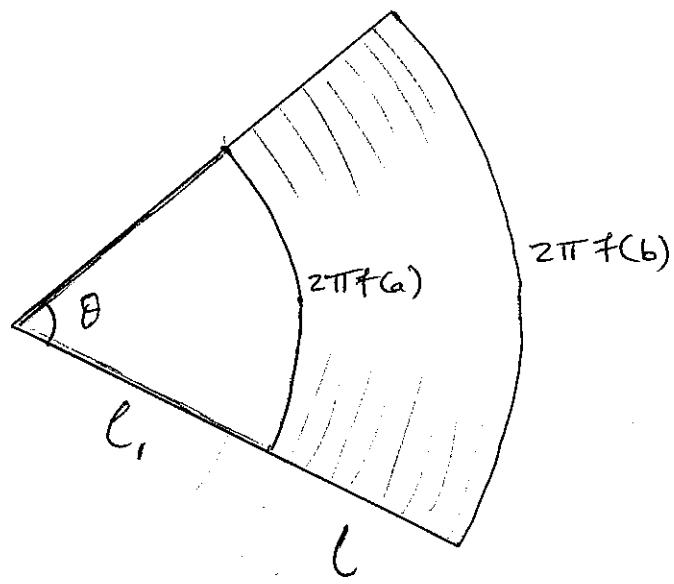
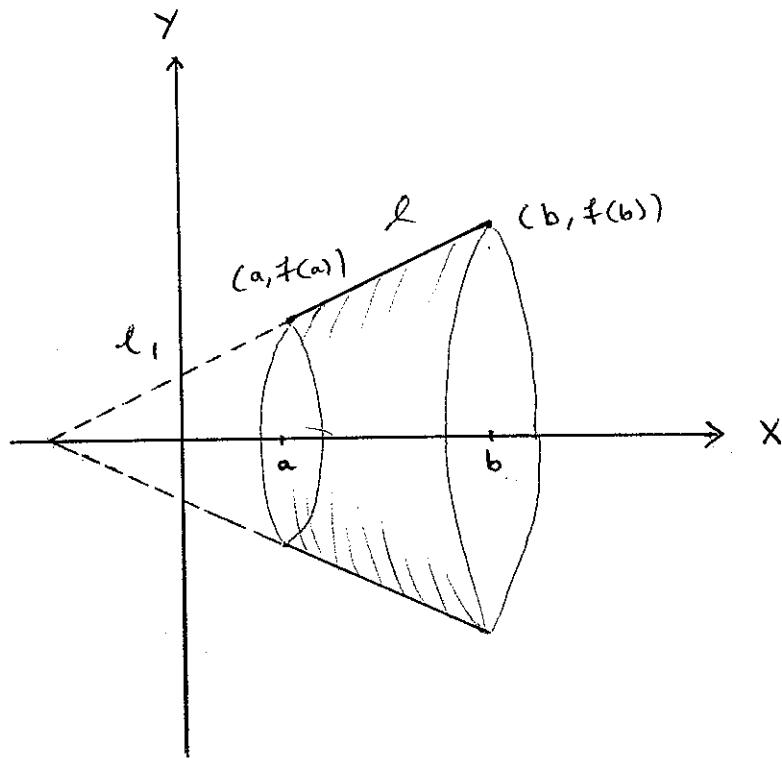
$y = \sqrt{1-x^2}$ $-1 \leq x \leq 1$ gives a sphere of radius 1.

Interesting question: What is the area of a surface of revolution.

We shall attack this question in a similar way to arc length. We will first consider the easiest example, a straight line, and then approximate the surface using rotated line segments. Finally we will take a limit and obtain a formula involving an integral.

Let $f(x)$ be a function whose graph is the straight line from $(a, f(a))$ to $(b, f(b))$. Let A denote the area of the surface of revolution coming from $f(x)$. From the picture on the next page we see that

$$A = \text{Area of large sector} - \text{Area of small sector.}$$



$$\text{Area of Large sector} = \frac{\theta}{2\pi} \times \underbrace{\pi(\ell_1 + \ell)^2}_{\text{area of large circle}}$$

$$\text{Area of Small sector} = \frac{\theta}{2\pi} \times \underbrace{\pi(\ell_1)^2}_{\text{area of small circle}}$$

Note that θ is in radians, hence by definition

$$\theta = \frac{2\pi f(a)}{\ell_1} = \frac{2\pi f(b)}{\ell_1 + \ell}$$

Inserting these two formulae into the above formulae gives

$$A = \pi f(b)(\ell_1 + \ell) - \pi f(a)\ell_1 = \pi(f(b) - f(a))\ell_1 + f(b)\ell$$

$$\text{Notice that } \frac{f(a)}{\ell_1} = \frac{f(b)}{\ell_1 + \ell} \Rightarrow (f(b) - f(a))\ell_1 = f(a)\ell$$

$$\text{Hence } A = \pi(f(a) + f(b))\ell.$$

Great! We've got a pretty easy answer for straight lines.

Let's do the general case.

Let's do the same as we did for arc length.

Let n be ~~a~~ whole number and divide the interval $[a, b]$ into n sub-intervals with endpoints x_0, x_1, \dots, x_n , each of length $\Delta x = \frac{b-a}{n}$.

For each i , with $0 \leq i \leq n$ let P_i be the point $(x_i, f(x_i))$. As in lecture 9 draw straight lines between successive P_i . Let S_i denote the surface of revolution coming from the straight line from P_{i-1} to P_i .

Over the page is a picture of this construction.

As for arc length we define the area of the surface of revolution of $f(x)$ to be

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n \text{Area}(S_i)$$

We want to make the sum a Riemann sum, then A will be a definite integral.

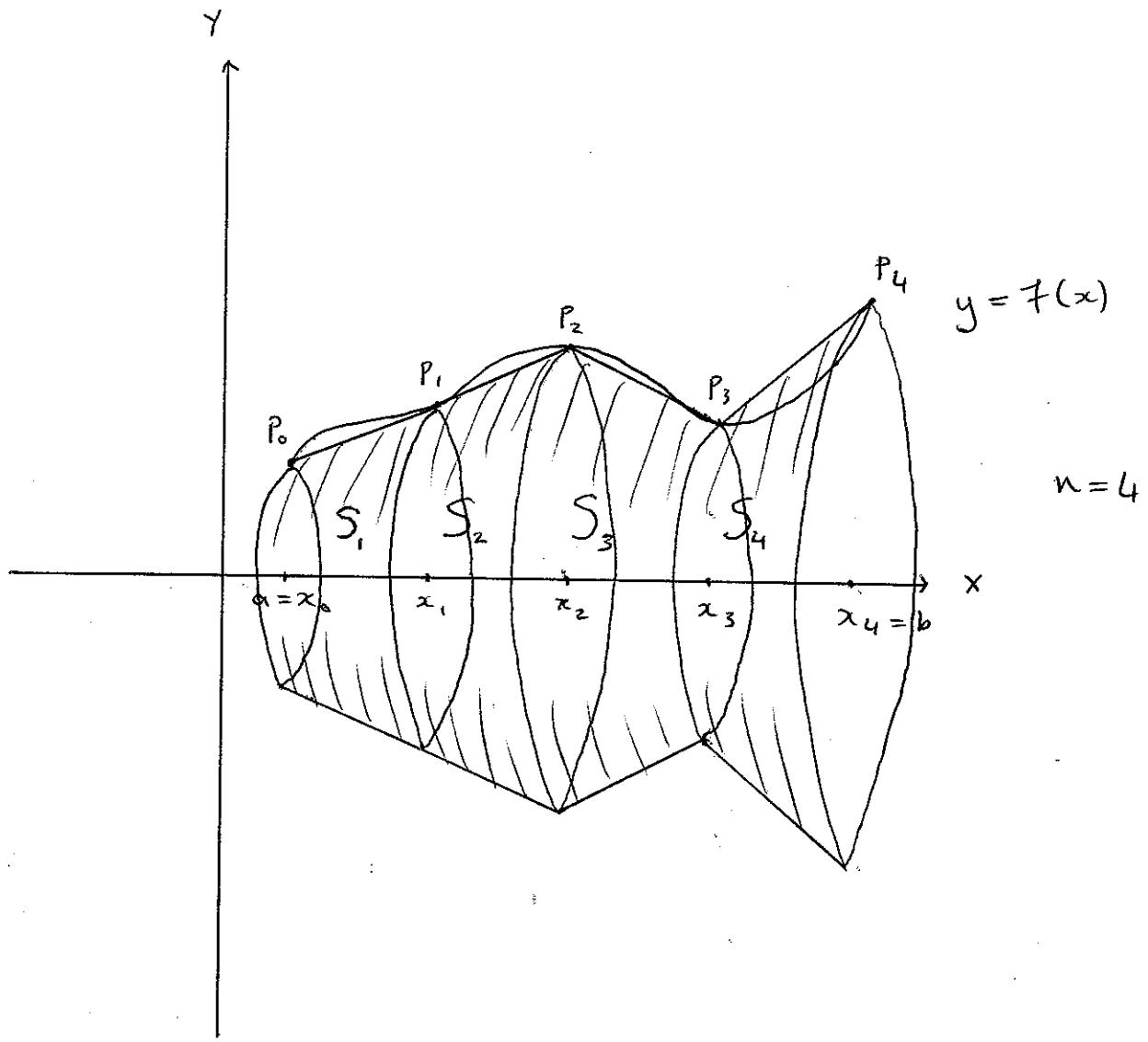
Let's analyse $\text{Area}(S_i)$. First recall that we can find a point x_i^* in the interval (x_{i-1}, x_i) such that

$$|P_{i-1}P_i| = \left(\sqrt{1 + f'(x_i^*)^2} \right) \Delta x.$$

By our previous work we know therefore that

$$\text{Area}(S_i) = \pi (f(x_{i-1}) + f(x_i)) |P_{i-1}P_i|$$

$$\begin{aligned} & \text{approximately} & & = 2\pi \left(\frac{f(x_{i-1}) + f(x_i)}{2} \right) \sqrt{1 + f'(x_i^*)^2} \Delta x \\ & \text{equal if } \Delta x \rightarrow & & \approx 2\pi f(x_i^*) \sqrt{1 + f'(x_i^*)^2} \Delta x \end{aligned}$$



Putting all of this together gives

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n \text{Area}(S_i) = \lim_{n \rightarrow \infty} \underbrace{\sum_{i=1}^n 2\pi f(x_i^*) \sqrt{1 + f'(x_i^*)^2} \Delta x}_{\text{Riemann sum for}} \\ = \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2} dx.$$

Awesome! We can now use integration techniques to determine A.

Example: $f(x) = \sqrt{1 - x^2} \quad -1 \leq x \leq 1.$

$$f'(x) = \frac{-2x}{2\sqrt{1-x^2}} \Rightarrow f'(x)^2 = \frac{x^2}{1-x^2} \Rightarrow$$

$$\sqrt{1 + f'(x)^2} = \sqrt{1 + \frac{x^2}{1-x^2}} = \sqrt{\frac{1-x^2}{1-x^2} + \frac{x^2}{1-x^2}} \\ = \sqrt{\frac{1}{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$$\text{Hence } A = 2\pi \int_{-1}^1 \sqrt{1-x^2} \cdot \frac{1}{\sqrt{1-x^2}} dx = 2\pi \int_{-1}^1 1 dx = 4\pi //$$

Hence the surface area of a sphere of radius 1 is 4π !