

Lecture 1 : Overview and RemindersWhat is Calculus?

Baby answer : "Calculus is the Mathematics of change!"

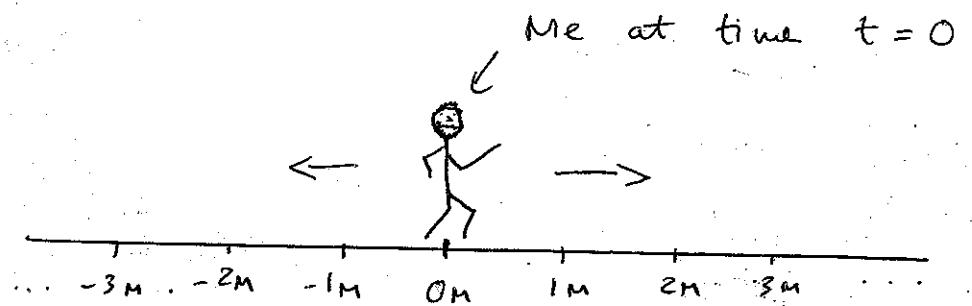
More grown up answer : "Calculus is the study of how different quantities vary with respect to each other".

Important real examples :

- 1) Population growth : How a population varies with respect to time.
- 2) Atmospheric pressure : How air pressure varies with respect to height from sea level.
- 3) Motion : How position in space varies with respect to time.

Let's look at this last example in more detail.

Let's mathematically model the simplest type of motion, namely motion in a straight line.

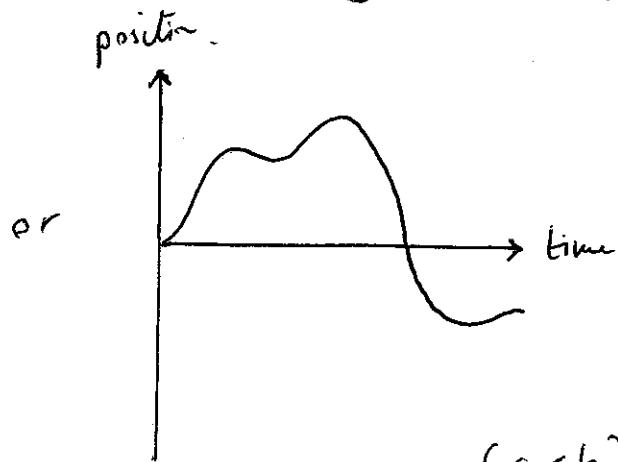
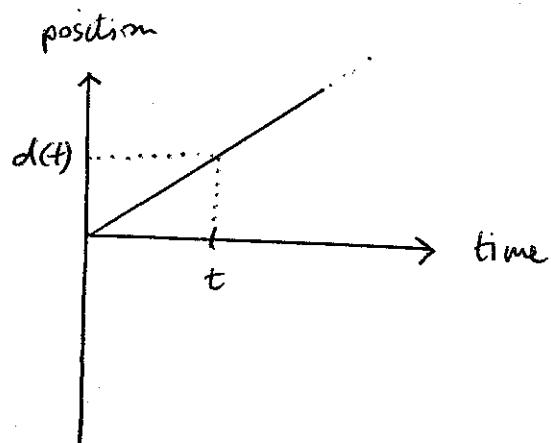


t = time

$d(t)$ = position at time t .

For simplicity assume $d(0) = 0$.

We can visualise my movement using a graph, e.g.



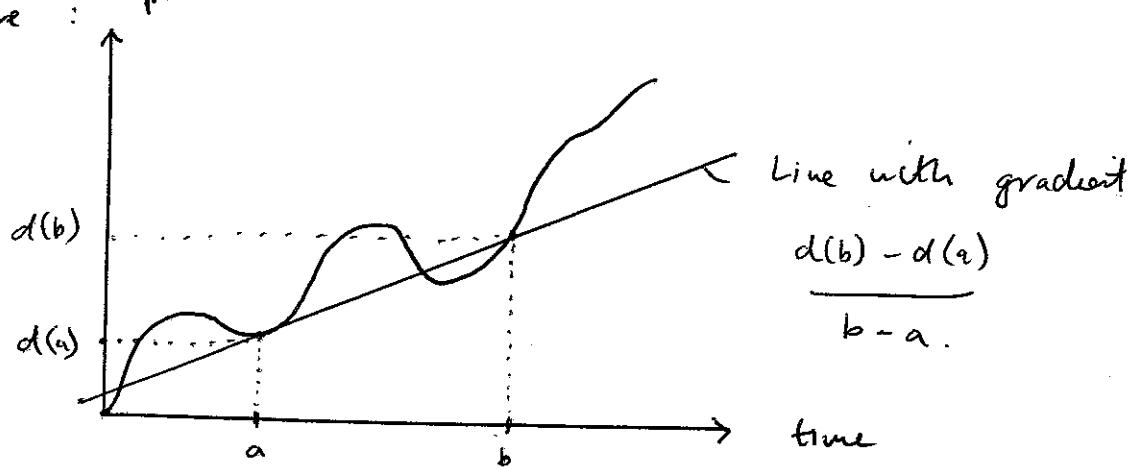
$$(a < b)$$

Interesting question : Between time $t=a$ and $t=b$,
what is my speed?

Most basic answer : Speed between $t=a$ and $t=b$ ($a < b$)

$$= \frac{\text{distance traveled}}{\text{time taken}} = \frac{d(b) - d(a)}{b - a}$$

Geometric picture :



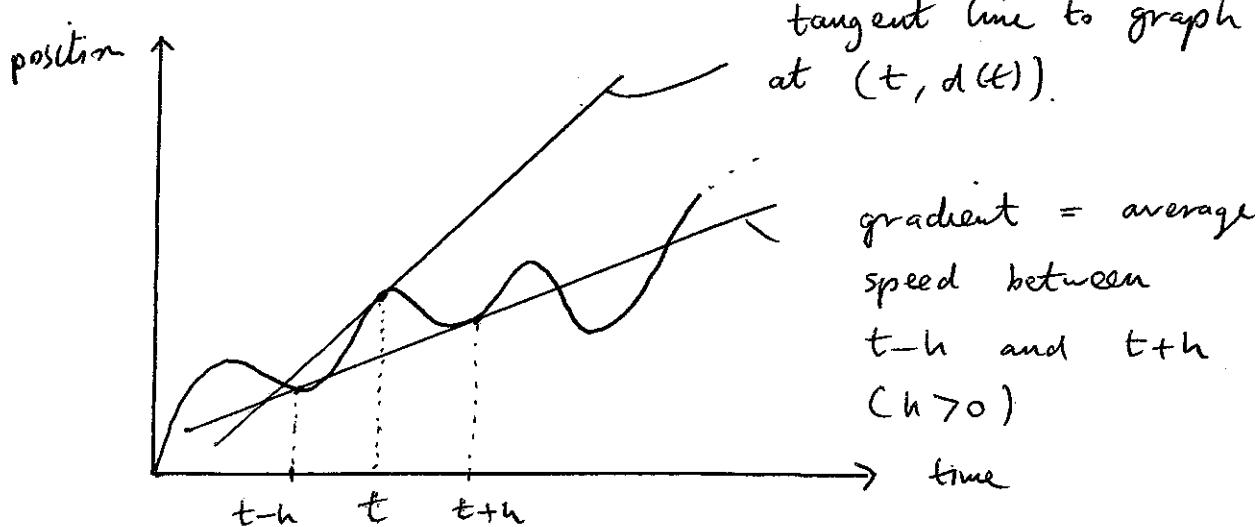
This loses lots of information : between $t=a$ and $t=b$ I might speed up or slow down in many different ways. $\frac{d(b) - d(a)}{b-a}$ tells me nothing about this. So we should really say

$$\text{Average speed between } t=a \text{ and } t=b = \frac{d(b) - d(a)}{b-a}.$$

More interesting question : What is my speed at a single given moment t ?

This is hard (the Greeks never managed to answer it!) because at a single moment we don't have an interval of time to measure distance travelled.

Clever Solution : Look at the average speed at smaller and smaller intervals of time around a given moment.



As h gets very small the straight line through $(t-h, d(t-h))$ and $(t+h, d(t+h))$ gets closer and closer to the tangent line. This leads to the following cool definition:

Speed at time t , = gradient of tangent line to denoted $d'(t)$ graph at $(t, d(t))$.

How do we work out this gradient? It's hard because we only know one point on line, namely $(t, d(t))$.

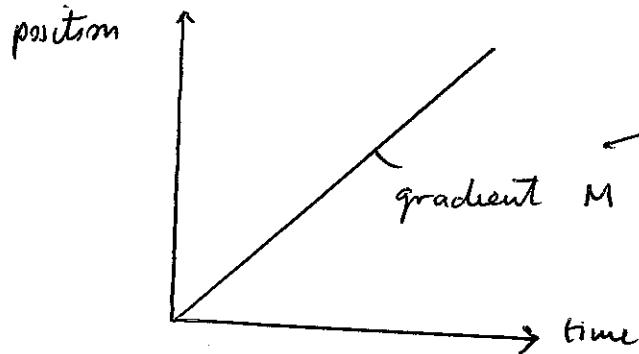
Mathematical solution: $d'(t) = \lim_{h \rightarrow 0_+} \frac{d(t+h) - d(t-h)}{2h}$ average speed

This means something $\rightarrow h \rightarrow 0_+$
 very precise mathematically
 that I won't cover now. See page 110 and 113.

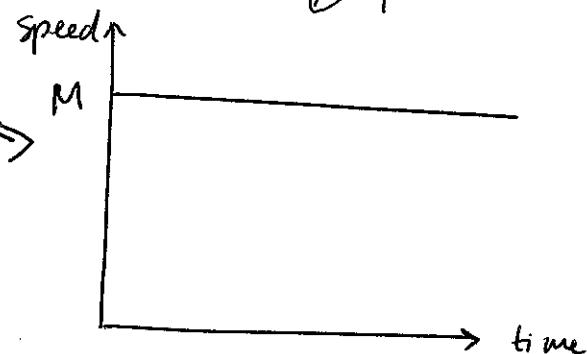
$$= \lim_{h \rightarrow 0} \frac{d(t+h) - d(t)}{h}$$

$d'(t)$, the speed at a single time t , gives us a new graph.

Easiest picture:



moving at constant
 ✓ speed, ie $s(t) = M$



Another interesting question : If we only know

the speed (let's call it $s(t)$) can we determine the distance travelled between time $t=a$ and $t=b$? "

Let's work backwards in our easiest picture.

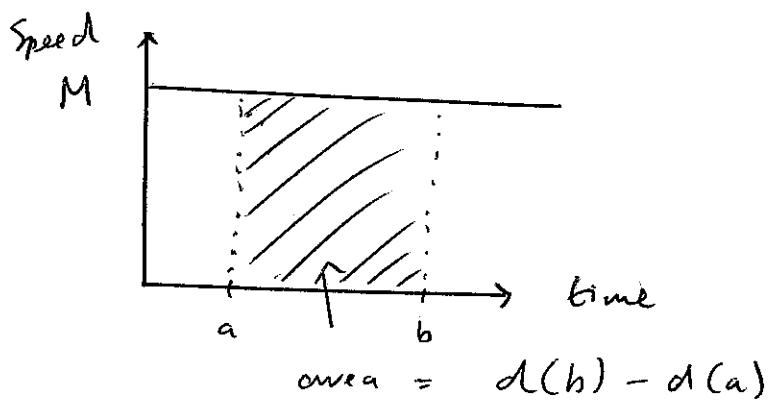
$$\text{distance travelled between } t=a \text{ and } t=b = d(b) - d(a)$$

$$= Mb - Ma$$

$$= M(b-a)$$

= Area under

speed graph
between $t=a$ and $t=b$



More generally, if we are given any graph for speed it is possible to show that the area under the graph between $t=a$ and $t=b$ must equal the distance travelled. We denote this area as follows :

$$\int_a^b s(t) dt = \text{Area under } s(t) \text{ between } t=a \text{ and } t=b.$$

Also observe the following :

$$\int_a^b d'(t) dt = d(b) - d(a)$$

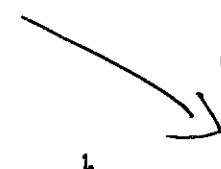
This example captures all the fundamental concepts of Calculus at a single variable.

Abstract reformulation

a "rule" which assigns to any number x , a unique number $f(x)$.

Main player : f , a function in a single variable, often denoted by x . (in our example $f = d$ and $x = t$)

 differentiation

 integration

$f'(x)$ - the derivative of $f(x)$.

$f'(x)$ = gradient of tangent to graph at $(x, f(x))$

$\int_a^b f(x) dx = \text{area under the graph between } x=a \text{ and } x=b.$

Fundamental Theorem of Calculus :

$$\int_a^b f'(x) dx = f(b) - f(a)$$

Even though differentiation and integration are defined very differently, the fundamental theorem provides a beautiful (and very useful) bridge between them.

Grown up answer to "What is Calculus?" :

Calculus is the study of single variable real valued functions in terms of differentiation and integration.