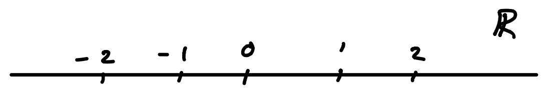
Vectors in $\mathbb{R}^n$

Aim: Develop a systematic approach to working with ordered sets of numbers.

$\mathbb{R}$ = Real numbers (Decimals)



n = a natural number $(1, 2, 3, 4, \dots)$

definition

$\mathbb{R}^n :=$ Set of ordered n-tuples of real numbers $x_1, x_2, \dots, x_n$.

Course Convention : Elements of $\mathbb{R}^n$ will always be written as a column.

$$\boxed{\mathbb{R}^n = \text{Set of all } \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ where } x_1, \dots, x_n \text{ are in } \mathbb{R}.}$$

We call an element of $\mathbb{R}^n$ a vector in $\mathbb{R}^n$.

$\mathbb{R}^1 = \mathbb{R}$ = number line

$\mathbb{R}^2$ = Plane $(x = x_1,\ y = x_2)$

$\mathbb{R}^3$ = 3D - space $(x = x_1,\ y = x_2,\ z = x_3)$

For $n > 3$ can't easily think about $\mathbb{R}^n$ geometrically.

Notation : $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ in $\mathbb{R}^n$

underline represents vector

$\underline{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ in $\mathbb{R}^n$

Called the zero vector in $\mathbb{R}^n$

Important Observations : Can "add" vectors in $\mathbb{R}^n$.
Can "scale" vectors in $\mathbb{R}^n$.

<u>Definition</u> : If $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$,

we define

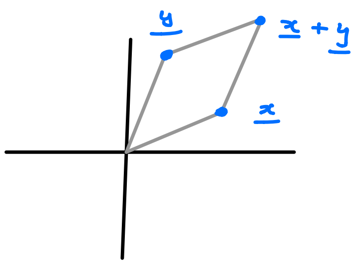
$$\underline{x} + \underline{y} := \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix} \quad \underline{\text{and}} \quad \lambda \underline{x} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}$$

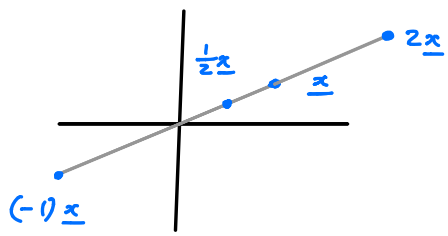
why its best to write as column

"sum" of $\underline{x}$ and $\underline{y}$

$\underline{x}$ "scaled" by λ

Geometric Picture in $\mathbb{R}^2$:





Same holds in $\mathbb{R}^3$.

Addition and scalar multiplication of vectors in $\mathbb{R}^n$ satisfy familiar rules of arithmetic. E.g. $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$, $\underline{0} + \underline{v} = \underline{v}$, $\lambda(\underline{u} + \underline{v}) = \lambda \underline{u} + \lambda \underline{v}$, $0\underline{v} = \underline{0}$. We write $-\underline{u} = (-1) \cdot \underline{u}$.

<u>Definition</u> Let $\underline{v}_1, \dots, \underline{v}_k$ be vectors in $\mathbb{R}^n$.

$\lambda_1 \underline{v}_1 + \lambda_2 \underline{v}_2 + \dots + \lambda_n \underline{v}_n$ = Linear combination of $\underline{v}_1, \dots, \underline{v}_n$

constants

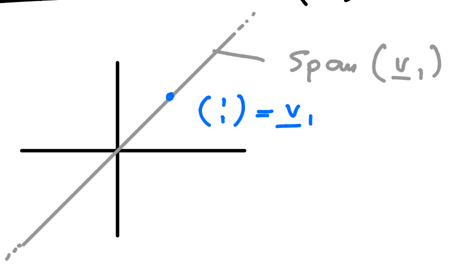
(its a vector in $\mathbb{R}^n$)

$\text{Span}(\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k)$ = All linear combinations of $\underline{v}_1, \dots, \underline{v}_n$

Intuition : $\text{Span}(\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k)$ = Everywhere we can get to in $\mathbb{R}^n$ by only travelling in directions $\underline{v}_1, \dots, \underline{v}_n$.

Note $\underline{0}$ is always in $\text{Span}(\underline{v}_1, \dots, \underline{v}_k)$.

Examples 1/ $\underline{v}_1 = \begin{pmatrix}1\\1\end{pmatrix}$ in $\mathbb{R}^2$ $\Rightarrow$ Span $(\underline{v}_1)$ = All $\lambda \underline{v}_1$ where λ in $\mathbb{R}$



2/ $\underline{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ in $\mathbb{R}^3$.

$$\lambda_1 \underline{v}_1 + \lambda_2 \underline{v}_2 = \lambda_1 \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda_2 \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}\lambda_1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\\lambda_2\\0\end{pmatrix} = \begin{pmatrix}\lambda_1\\\lambda_2\\0\end{pmatrix}$$

$\Rightarrow$ Span $(\underline{v}_1, \underline{v}_2)$ = x-y plane inside 3D-space

3/ $\underline{v}_1 = \begin{pmatrix}2\\-1\\4\end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix}-1\\\frac{1}{2}\\-2\end{pmatrix}$ in $\mathbb{R}^3$

Note that $-2\underline{v}_2 = \underline{v}_1$ $\Rightarrow$ $\lambda_1 \underline{v}_1 + \lambda_2 \underline{v}_2 = \lambda_1(-2\underline{v}_2) + \lambda_2 \underline{v}_2$

$$= \underbrace{(-2\lambda_1 + \lambda_2)}_{\text{arbitrary constant}} \underline{v}_2$$

$\Rightarrow$ Span $(\underline{v}_1, \underline{v}_2)$ = Span$(\underline{v}_2)$ = Straight line through $\underline{0}$ and $\begin{pmatrix}-1\\1/2\\-2\end{pmatrix}$ in $\mathbb{R}^3$.

Conclusions (in $\mathbb{R}^2/\mathbb{R}^3$) : $\underline{v}_1 \neq \underline{0}$ $\Rightarrow$ Span $(\underline{v}_1)$ = straight line through $\underline{0}$ and $\underline{v}_1$

$\underline{v}_1 \neq \underline{0}$, $\underline{v}_2 \neq \underline{0}$ $\Rightarrow$ Span$(\underline{v}_1, \underline{v}_2)$ = $\begin{cases} \text{Plane containing } \underline{0}, \underline{v}_1 \text{ and } \underline{v}_2 & \text{if } \underline{v}_1, \underline{v}_2 \text{ in independent directions} \\ \text{Line containing } \underline{0}, \underline{v}_1, \underline{v}_2 & \text{if } \underline{v}_1, \underline{v}_2 \text{ not in independent directions} \end{cases}$

Nice Exercise : What are the possible for Span $(\underline{v}_1, \underline{v}_2, \underline{v}_3)$?

Key Take Aways : Given $\underline{v}_1, \dots, \underline{v}_k$ in $\mathbb{R}^n$

$\text{Span}(\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k)$ = Everywhere we can get to in $\mathbb{R}^n$ travelling only in directions $\underline{v}_1, \dots, \underline{v}_k$.

Depending on positions some vectors may be unnecessary.