Vectors and Uniqueness of Linear System Solutions

Q/ If $A\underline{x} = \underline{b}$ admits a solution what properties of A determine if the solution is unique?

Related Problem: Homogeneous Linear System : $A\underline{x} = \underline{0}$

$$A\underline{x} = \underline{0} \iff x_1\underline{a}_1 + x_2\underline{a}_2 + \dots + x_n\underline{a}_n = \underline{0}$$

Observe $\underline{x} = \underline{0}$ is always a solution.

Definition Let $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k$ be vectors in $\mathbb{R}^m$.
We say $\{\underline{v}_1, \dots, \underline{v}_k\}$ is linearly independent if

$$\lambda_1\underline{v}_1 + \lambda_2\underline{v}_2 + \dots + \lambda_k\underline{v}_k = \underline{0} \Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0$$

If not we say $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k\}$ is linearly dependent.

Remark

$\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k\}$ Linearly dependent $\iff$ There exist $\lambda_1, \dots, \lambda_k$ not all zero such that $\lambda_1\underline{v}_1 + \dots + \lambda_k\underline{v}_k = \underline{0}$

E.g. If $\lambda_1 \neq 0 \Rightarrow \underline{v}_1 + \frac{\lambda_2}{\lambda_1}\underline{v}_2 + \dots + \frac{\lambda_k}{\lambda_1}\underline{v}_k = 0$

$$\Rightarrow \underline{v}_1 = \left(\frac{-\lambda_2}{\lambda_1}\right)\underline{v}_2 + \dots + \left(\frac{-\lambda_k}{\lambda_1}\right)\underline{v}_k$$

$\Rightarrow \underline{v}_1$ is in Span $(\underline{v}_2, \underline{v}_3, \dots, \underline{v}_k)$

Example

$$10\begin{pmatrix}1\\2\\3\end{pmatrix} - 5\begin{pmatrix}4\\5\\6\end{pmatrix} + 5\begin{pmatrix}2\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

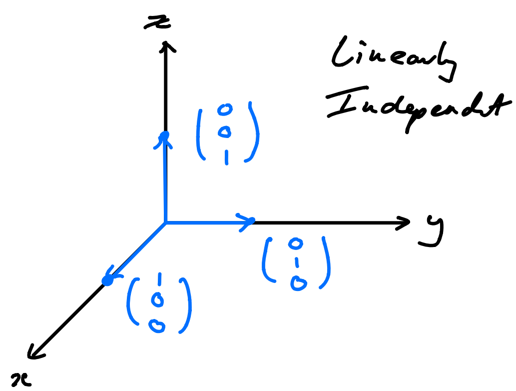
$$\Rightarrow \begin{pmatrix}1\\2\\3\end{pmatrix} \text{ in } \mathrm{Span}\left(\begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}\right)$$

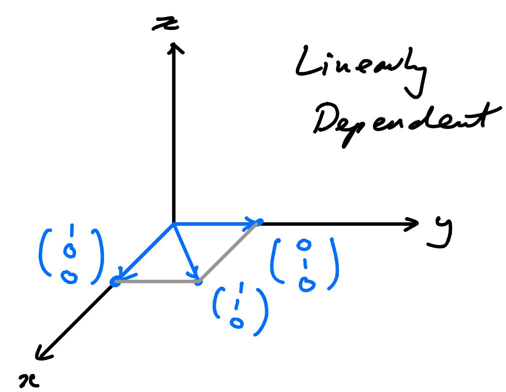
Intuition: $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ Linearly independent

$\Longleftrightarrow$ $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ are in completely independent directions.

E.g. $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ Linearly independent

$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$ Linearly dependent





Conclusion

$$\boxed{\begin{array}{lcl} A\underline{x} = \underline{0} \text{ has unique solution } (\text{ie } \underline{x} = \underline{0}) & \Longleftrightarrow & \text{Columns of } A \text{ are Linearly independent} \\ \Updownarrow & & \\ \text{Reduced } (A \mid \underline{0}) \text{ has no free columns} & & \end{array}}$$

Nice Consequence : $\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n$ vectors in $\mathbb{R}^m$ $(A = (\underline{a}_1 \cdots \underline{a}_n))$

$m < n \Rightarrow$ Reduced $(A \mid \underline{0})$ has free columns $\Rightarrow \{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\}$ **Linearly dependent**

Makes sense : In $\mathbb{R}^n$ there can't be more than n independent directions intuitively.

Examples : There are not 3 independent directions in $\mathbb{R}^2$.

Back to non-homogeneous case : $A\underline{x} = \underline{b}$

$\underline{v}_p$ = particular solution to $A\underline{x} = \underline{b}$ (i.e. $A\underline{v}_p = \underline{b}$)

solution to homogeneous system

$A\underline{v} = \underline{b} \Rightarrow A(\underline{v} - \underline{v}_p) = A\underline{v} - A\underline{v}_p = \underline{b} - \underline{b} = \underline{0}$

$\Rightarrow$ General Solution to $A\underline{x} = \underline{b}$ is $\underline{v}_p + \underline{v}_h$, where $\underline{v}_h$ = general solution to $A\underline{x} = \underline{0}$

Conclusion

$A\underline{x} = \underline{b}$ has unique solution $\iff$ $\underline{b}$ in Span$(\underline{a}_1, \dots, \underline{a}_n)$ **and** Columns of A Linearly independent

$\Updownarrow$

Last column of reduced $(A \mid \underline{b})$ not a pivot **and** No free columns

Example

$$\begin{array}{c} \begin{array}{ccc|c} & A & & b \end{array} \\ \left(\begin{array}{ccc|c} 1 & 3 & 5 & 1 \\ 0 & -2 & 2 & 1 \end{array}\right) \end{array} \Rightarrow \begin{aligned} x_1 + 3x_2 + 5x_3 &= 1 \\ -2x_2 + 2x_3 &= 1 \end{aligned}$$

$\uparrow$ free ($\Rightarrow$ columns of A linearly dependent)

$$\Rightarrow \quad x_2 = x_3 - \frac{1}{2}$$

$$x_1 = \frac{5}{2} - 8x_3$$

$\Rightarrow$ General Solution to $A\underline{x} = \underline{b}$

$$= \begin{pmatrix} \frac{5}{2} - 8x_3 \\ x_3 - \frac{1}{2} \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{5}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}}_{= \underline{v}_p} + x_3 \underbrace{\begin{pmatrix} -8 \\ 1 \\ 1 \end{pmatrix}}_{= \underline{v}_n}$$

Overview

$(A = (\underline{a}_1\ \underline{a}_2 \cdots \underline{a}_n))$

$A\underline{x} = \underline{b}$ has no solutions $\Longleftrightarrow$ $\underline{b}$ $\underline{\text{not}}$ in Span$(\underline{a}_1, \ldots, \underline{a}_n)$

$A\underline{x} = \underline{b}$ admits unique solution $\Longleftrightarrow$ $\underline{b}$ in Span$(\underline{a}_1, \ldots, \underline{a}_n)$ $\underline{\text{and}}$ $\{\underline{a}_1, \ldots, \underline{a}_n\}$ Linearly independent.

$A\underline{x} = \underline{b}$ admits infinitely many solutions $\Longleftrightarrow$ $\underline{b}$ in Span$(\underline{a}_1, \ldots, \underline{a}_n)$ $\underline{\text{and}}$ $\{\underline{a}_1, \ldots, \underline{a}_n\}$ Linearly dependent.