Vectors and Existence of Linear System Solutions

Let a linear system (m equations in n variables) have augmented matrix

vector notation   vectors in $\mathbb{R}^m$

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right) = \left(\underbrace{\underline{a}_1 \ \underline{a}_2 \ \cdots \ \underline{a}_n}_{\substack{\text{Coefficient matrix} \\ A}} \,\middle|\, \underline{b}\right)$$

Important Observation

$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ a solution of linear system $(A \,|\, \underline{b})$

$$\Longleftrightarrow \quad \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + x_{mn}x_n = b_m \end{array}$$

$$\Longleftrightarrow \quad x_1 \underbrace{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}}_{= \underline{a}_1} + x_2 \underbrace{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}}_{= \underline{a}_2} + \cdots + x_n \underbrace{\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}}_{= \underline{a}_n} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{= \underline{b}}$$

$\Rightarrow \quad \underline{b}$ in $\text{Span}(\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n)$

each column vector in $\mathbb{R}^m$

Matrix Equation notation : If $A = (\underline{a}_1 \cdots \underline{a}_n)$ (Say A is $m \times n$ matrix)

and $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ in $\mathbb{R}^n$ then

linear combination in $\mathbb{R}^m$.

$$A\underline{x} := x_1\underline{a}_1 + x_2\underline{a}_2 + \cdots + x_n\underline{a}_n$$

Conclusion

The linear system $(A \mid \underline{b})$ admits a solution $\iff$ $A\underline{x} = \underline{b}$ for some $\underline{x}$ in $\mathbb{R}^n$ $\iff$ $\underline{b}$ in $\mathrm{Span}(\underline{a}_1, \dots, \underline{a}_n)$

$\Updownarrow$

Last column of reduced augmented matrix is **not** a pivot column

Example Is $\begin{pmatrix}1\\2\\3\end{pmatrix}$ in $\mathrm{Span}\left(\begin{pmatrix}-1\\1\\2\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}\right)$?

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \text{ in } \mathrm{Span}\left(\begin{pmatrix}-1\\1\\2\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}\right) \iff \left(\begin{array}{ccc|c}-1 & 2 & 1 & 1\\ 1 & 1 & 2 & 2\\ 2 & 0 & 2 & 3\end{array}\right) \text{ consistent}$$

$$\left(\begin{array}{ccc|c}-1 & 2 & 1 & 1\\ 1 & 1 & 2 & 2\\ 2 & 0 & 2 & 3\end{array}\right) \longrightarrow \left(\begin{array}{ccc|c}1 & -2 & -1 & -1\\ 1 & 1 & 2 & 2\\ 2 & 0 & 2 & 3\end{array}\right) \longrightarrow \left(\begin{array}{ccc|c}1 & -2 & -1 & -1\\ 0 & 3 & 3 & 3\\ 0 & 4 & 4 & 5\end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c}1 & -2 & -1 & -1\\ 0 & 1 & 1 & 1\\ 0 & 4 & 4 & 5\end{array}\right) \longrightarrow \left(\begin{array}{ccc|c}1 & -2 & -1 & -1\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 1\end{array}\right)$$

pivot

System is Inconsistent $\Leftarrow$

$\Downarrow$

$\begin{pmatrix}1\\2\\3\end{pmatrix}$ <u>not in</u> $\mathrm{Span}\left(\begin{pmatrix}-1\\1\\2\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}\right)$

Linear System $(A \mid \underline{b})$ has solution for <u>any</u> $\underline{b}$ $\iff$ $\mathrm{Span}(\underline{a}_1, \dots, \underline{a}_n) = \mathbb{R}^m$

$\Updownarrow$

Last column of reduced augmented matrix is never a pivot $\iff$ <u>Reduced A has no zero rows</u>

<u>Example</u> Is it true that $\mathrm{Span}\left(\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\-1\\1\end{pmatrix}\right) = \mathbb{R}^3$

$$\begin{pmatrix}1 & 1 & 2\\2 & 0 & -1\\3 & 1 & 1\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 1 & 2\\0 & -2 & -5\\0 & -2 & -5\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 1 & 2\\0 & -2 & -5\\0 & 0 & 0\end{pmatrix} \leftarrow \text{zero row}$$

$$\Rightarrow \quad \mathrm{Span}\left(\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\-1\\1\end{pmatrix}\right) \neq \mathbb{R}^3 .$$