Vector Spaces and Linear Transformations

Q: What does a matrix and differentiation have in common?

A – $m \times n$ matrix

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$\underline{x} \longmapsto A\underline{x}$$

1/ $T_A(\underline{u}+\underline{v}) = T_A(\underline{u}) + T_A(\underline{v})$

2/ $T_A(\lambda \underline{u}) = \lambda T_A(\underline{u})$

$\mathbb{C}'(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \text{ differentiable with continuous derivative} \right\}$

$\mathbb{C}(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \text{ continuous} \right\}$

$$\frac{d}{dx} : \mathbb{C}'(\mathbb{R}) \longrightarrow \mathbb{C}(\mathbb{R})$$
$$f \longmapsto \frac{d}{dx}(f)$$

1/ (Sum Rule)

sum of functions

$$\frac{d}{dx}(f+g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)$$

2/ (Constant Multiple Rule)

$$\frac{d}{dx}(\lambda f) = \lambda \frac{d}{dx}(f)$$

Linearity Conditions

<u>Important Observation</u>: 1/ and 2/ make sense because we can "add" and "scalar multiply" in $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{C}'(\mathbb{R})$ and $\mathbb{C}(\mathbb{R})$.

<u>Intuitive Definition</u>: A (real) <u>vector space</u> is a set V that comes with a concept of "addition" and "real scalar multiplication" satisfying nice properties.

Vector spaces are everywhere!

Examples

1/ $\mathbb{R}$ with usual addition and multiplication.

2/ $\mathbb{R}^n$ with usual addition and scalar multiplication

3/ $\mathbb{R}$-valued functions on a fixed set with usual sum and scalar multiplication of functions.

$\{f : \mathbb{R} \to \mathbb{R}\}$, $\{f : [a,b] \longrightarrow \mathbb{R}\}$

4/ $C^1(\mathbb{R})$, $C(\mathbb{R})$, $C^\infty(\mathbb{R})$ ⟵ infinitely differentiable functions on $\mathbb{R}$

5/ $\{f : [a,b] \to \mathbb{R}$, Integrable$\}$

6/ Sequences of real numbers

7/ Convergent infinite series of real numbers

8/ Polynomials with real coefficients

9/ Power series with real coefficients

10/ $\mathbb{C}$ - complex numbers

11/ Planes and lines containing origin in $\mathbb{R}^3$.

12/ Random variables on sample space

13/ Solutions to homogeneous differential equation $ay'' + by' + cy = 0$

A <u>vector</u> in a vector space V is just an element of V

E.g. $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a vector in $\mathbb{R}^3$, $\sin(x)$ is a vector in $C(\mathbb{R})$. We always use underline notation for vectors

Observation : In all cases, addition and scalar multiplication obey common rules : For all $\underline{u}, \underline{v}, \underline{w}$ in V and λ, μ in $\mathbb{R}$

1/ $(\underline{u}+\underline{v})+\underline{w} = \underline{u}+(\underline{v}+\underline{w})$

2/ There is $\underline{0}$ such that $\underline{0}+\underline{v} = \underline{v}+\underline{0} = \underline{v}$

3/ Given $\underline{v}$ there is $-\underline{v}$ in V such that
$\underline{v}+(-\underline{v}) = (-\underline{v})+\underline{v} = \underline{0}$

4/ $\underline{u}+\underline{v} = \underline{v}+\underline{u}$

5/ $\lambda(\underline{u}+\underline{v}) = \lambda\underline{u}+\lambda\underline{v}$

6/ $(\lambda+\mu)\underline{u} = \lambda\underline{u}+\mu\underline{u}$

7/ $(\lambda\mu)\underline{u} = \lambda(\mu\underline{u})$

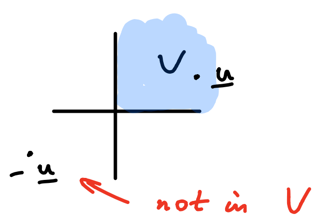
8/ $1.\underline{u} = \underline{u}$

Consequences : $(-1)\underline{v} = -\underline{v}$, $0\underline{v} = \underline{0}$, $\lambda\underline{0} = \underline{0}$

<u>Precise Definition</u> : A (real) vector space is a set which comes with an "addition" and "scalar multiplication" which satisfy all 8 above properties.

real

<u>Non-example</u> V = Upper right quadrant in $\mathbb{R}^2$



<u>Important Definition</u> A linear transformation between two vector spaces V and W is a function $T : V \longrightarrow W$ such that

in V   in W

1/ $T(\underline{u}+\underline{v}) = T(\underline{u})+T(\underline{v})$ for all $\underline{u},\underline{v}$ in V

in V   in W

2/ $T(\lambda\underline{u}) = \lambda T(\underline{u})$ for all $\underline{u}$ in V and λ in $\mathbb{R}$

<u>Examples</u>

1/ A – $m\times n$ matrix $\Rightarrow$ T_A = Linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

2/ $\frac{d}{dx}$ = Linear transformation from $\mathbb{C}^1(\mathbb{R})$ to $\mathbb{C}(\mathbb{R})$

3/ $\int_a^b$ = Linear transformation from $\{f : [a,b] \to \mathbb{R}, \text{Integrable}\}$ to $\mathbb{R}$.

4/ $\mathbb{E}$ = Linear transformation from {Random Variables} to $\mathbb{R}$

expected value

Linear Algebra = Study of Linear Transformations between vector spaces.

Remark Some examples are naturally subsets of each other. E.g. $\mathbb{C}^1(\mathbb{R}) \subset \mathbb{C}(\mathbb{R})$. Planes/Lines containing origin in $\mathbb{R}^3$.