Teaching Portfolio

Alexander Paulin

Department of Mathematics, King's College London

Contents

Teaching Philosophy

Mathematics is one of the great collective human endeavours and good teaching is fundamental to its development. This is the case throughout all mathematics, from elementary school to pure research. Significant advances are often made by individuals, but it is through our joint efforts that the subject flourishes. If Euclid had gained his insight into geometry and been content to share it with no one, mathematics would likely be unrecognizable today.

Learning mathematics is hard. To the uninitiated it is veiled in mystery, with a seemingly impenetrable language all of its own. In fact, mathematics is deeply intuitive, built from our most basic observations about the behaviour of the natural world. Being able to see past the technicalities of the language to the underlying foundational concepts is one of the key skills required to master the subject. This is why good tuition is so vital: the teacher acts as a guide, placing ideas in context, gradually revealing the elegant core principles. Truly great teachers give the student the sense that they are discovering the subject for themselves.

We all begin teaching with the experience of being taught. As a student, one comes to recognize both good and bad tuition. My first experience of good tuition was during my Cambridge entrance interview. After the formal interview, the professor took extra time to explain how what I had just seen could be used to prove the irrationality of π. This small moment had a big impact on me: that someone so advanced had taken the time to develop my understanding gave me confidence in my own abilities. At the same time, his enthusiasm renewed my own. This experience has gone on to form the backbone of my approach to teaching: always be generous to students and let them see how passionate you are about the subject. If a student is struggling, office hours should go on until they understand and not when the bell rings. If a student wants to see how a concept develops, take time to show them.

In the years following this experience I have learned much, both as a student and a teacher, about what good tuition means. The following core principles form the foundation of my own teaching philosophy:

- Be able to lucidly explain new and difficult ideas
- Have a good relationship with the class
- Be organised and well prepared
- Show enthusiasm for the subject
- Be approachable and available
- Be patient, setting an appropriate pace and difficulty
- Actively make efforts to improve student confidence

Teaching Strategy

Good teaching is about effectively putting these principles into practice. How one does this depends on the audience. For example, supervising a senior thesis is very different than teaching a large undergraduate class. Regardless of the situation, it is vital to be thoroughly prepared before seeing students. As such, when developing a new course I first assess the academic background of the class before constructing a syllabus and writing lectures. For example, before I started teaching linear algebra at King's College I spoke to other professors about the content of the prerequisite courses. In all my courses, I make sure electronic notes are available before each lecture so students have a comprehensive exposition of the material from the outset. In smaller classes I hand out printed copies of these notes at the beginning of each lecture. Students have responded very positively to this as it gives them the core material but also the opportunity to embellish it with extra information. As a result, every student in the class ends up with a clear and highly detailed set of lecture notes.

When introducing a new concept, simplicity is key. I make efforts to keep explanations concise and always provide simple but instructive examples. For example, when introducing cosets of a subgroup, I begin by considering translations of a line in the plane. This sets the subject within a coherent narrative and gives the student a clear picture from which they can start to build their intuition.

At the beginning of any course it is important to state the goals of the class as well as the commitment expected from the students. For this reason I use the first lecture to outline the syllabus, and give a motivating overview of the subject. I also stress that a difficult problem set can take perhaps 10 or more hours of serious work to complete. This gives students a realistic understanding of how much time they need to invest in the material.

When teaching, it is important to continuously reassess the progress of your students as well as your own performance. This is why a good, open dialogue with your class is vital. Getting students to actively interact during lectures is the best way to achieve this. This is especially challenging in large lectures, where students naturally feel more intimidated. I've found that a good way to solve this problem is by asking well chosen questions after introducing new concepts. This helps to get the students talking and over time they become confident enough to ask questions themselves. I have also found that handing out an informal teaching evaluation questionnaire in the first few weeks of a course gives valuable feedback about the how to improve lectures in real time. For example, in my recent linear algebra class, I gave extended review lectures before each of the three midterms, responding to requests from the class.

In large classes, lectures are often supplemented by weekly, graduate lead tutorials. I play an active role in these, frequently talking to the graduate students who run them. This is very important as it gives another way to gauge the class' progress. I periodically attend the tutorials myself, as some students are nervous about coming to office hours and this is a good way of speaking to them face to face and building their confidence.

A challenge inherent to all forms of teaching is dealing with a wide range of abilities. In any class it is inevitable that some students will struggle. It is important to provide these students with the support they need, without neglecting the more able students. A good solution I've found is two-fold: provide a wide range of problems of varying difficulty and be available whenever they ask for help. The people who are struggling will be able to find a foothold in the easier exercises and slowly build their understanding; the best students will be challenged by the more difficult exercises. If a student has spent time on a problem but been unable to make progress, I encourage them to come and speak to me. I always take the time to explain the solution to every problem I am asked about. In general, I encourage as many of my students as possible to come to my office hours.

Finally I feel that it is important to share with the students my own love of the subject. Mathematics is beautiful and mysterious, with so much still to be discovered. This attitude is reflected in the way I teach. I always take time to expand on the deeper aspects of any subject I'm asked about. Many of my best students have come back to do independent study courses motivated by these experiences.

Teaching Experience

In the six years since I completed my PhD, my teaching responsibilities have largely focused on undergraduate and graduate mathematics.

I have taught the following undergraduate courses:

- Honours Linear Algebra and Differential Equations
  UC Berkeley, Lower Division, 4 credits, 30-40 students
- Abstract Algebra
  UC Berkeley, Upper Division, 4 credits, 30-40 students, taught 5 classes
- Honours Multivariable Calculus
  UC Berkeley, Lower Division, 4 credits, 30-40 students
- Linear Algebra
  Nottingham and King's College London, 200-300 students, taught 3 classes
- Foundations of Mathematics for Biology and Chemistry Majors
  Nottingham, 20-30 students
- Rings and Modules
  King's College London, 50-60 students, taught 2 classes

I have taught the following graduate courses:

- Algebraic Number Theory and the Langlands Program
  UC Berkeley, 10-20 students
- Modular Forms
  UC Berkeley, 10-20 students
- Introduction to the Geometric Langlands Program
  Nottingham, 5-10 students

I have supervised the following senior undergraduate theses:

- Derived Categories
- Complex Semi-Simple Lie Algebras
- Differential Equations and Stokes Theorem
- The Etale Fundamental Group
- The *p*-adic Number
- Central Simple Algebras and the Brauer Group
- Introduction to Category Theory

I have supervised the following independent undergraduate and graduate reading courses in the following topics:

- Elliptic Curves
- *p*-adic Hodge Theory
- *p*-adic Modular Forms
- Etale Cohomology
- Automorphic Representations
- Smooth Representation Theory of *p*-adic Reductive Groups
- Commutative Algebra

I have organised and taught in the following research study group:

- Geometric Langlands and Functoriality,
  King's College London

I have taught numerous study group lectures in many different research topics at Imperial College, UCL, King's College London and UC Berkeley.

Course Syllabi, Lectures, Assignments, Examinations

Designing courses in a comprehensive and methodical way is vital to the teaching process. I always provide comprehensive, well written lecture notes to all my classes. I have included representative samples of the teaching materials I have developed over the last six years. More precisely, I have included the following in the appendices:

- Syllabus for the abstract algebra class I taught at UC Berkeley, including the introductory motivation to the course. (**see Appendix A**)
- A lecture from my abstract algebra class where I introduce the concept of a group. (**see Appendix B**)
- Homework exercises and solutions from my current linear algebra class. (**see Appendix C)**
- Midterm and final exams for my current linear algebra class. (**see Appendix D)**

Teaching Evaluations

I constantly strive to improve my teaching and student evaluations are an integral part of this process. The following student evaluations are from numerous courses I have taught over the last six years.

Mean student ratings on a 4 point scale: 0 (strongly disagree) to 4 (strongly agree)
King's College 2011/2012, 226 undergraduate students

3.78	Lecturer is audible	**3.58**	Lectures are well organised
3.48	Lecturer explains the material clearly	**3.58**	Lecturer has a good relationship with the class
3.02	Gives lectures in a stimulating manner	**3.41**	Lecturer is available outside of class

Mean student ratings on a 7 point scale: 0 (lecturer was extremely ineffective) to 7 (lecture was extremely effective)
UC Berkeley 2008 to 2011

6.4	Abstract Algebra, 200 undergraduate students
6.3	Number Theory, 30 graduate students
6.1	Multivariable Calculus, 20 undergraduate students

Anonymous student evaluations also give the students the opportunity to provide specific comments, allowing me to strengthen my teaching. The following are representative examples from the last six years:

> *"Best math professor I have had at Berkeley. His presentation skills, insight and quick thinking are amazing. But what really sets him apart is his ability to work one on one with students. He takes time out of his schedule to help students understand the material and has tremendous patience. His homework assignments are fun too!"* - Undergraduate student, Abstract Algebra, 2009 (UC Berkeley)

> *"Professor Paulin is literally one of the best math professors Berkeley has ever seen. I feel so privileged to have taken abstract algebra with him. I was unsure if math was the right major for me, but because of him, I see how beautiful math can really be."* - Undergraduate student, Abstract Algebra, 2011 (UC Berkeley)

> *"Professor Paulin is an excellent teacher. He is very willing to help students with any questions they have. In addition he presents material in a way that makes it very interesting, demonstrating why the theorems work. I really appreciate that he gave us notes so we didn't have to spend the whole class writing."* - Undergraduate student, Multivariable Calculus, 2010 (UC Berkeley)

> *"He's very precise when answering questions and really seems like he wants us to learn."* Undergraduate student, Multivariable Calculus, 2010 (UC Berkeley)

Advising

In addition to regular office hours, I tell all my students that I am free to meet them whenever they need help. This one on one interaction is a great way to build a good relationship with the class. I also feel it is important to provide pastoral care for students when they need it. For example, while at Nottingham I was the personal tutor of 20 students, with whom I met several times each semester.

Many of the students who regularly come to my office hours have gone on to do senior theses and independent reading courses with me. I deeply enjoy this form of teaching and I am always happy to help my students pursue their interests. Numerous of my best students have gone on to graduate school. For example, one of my students at UC Berkeley asked me about studying mathematics in Cambridge, where I had been an undergraduate. I encouraged and supported his application and after he was accepted we spent time deciding which courses he should take. He is currently a graduate student at the University of Texas, Austin. Seeing the evolution of such students over time is perhaps the most rewarding aspect of teaching.

Teaching and Technology

I am very interested in enhancing the process of learning mathematics using technology. Mathematics is inherently a hierarchical subject: each concept has a hierarchy of concepts behind it. For example, to understand the definition of a ring one must understand the definition of an Abelian group, and to understand this one must understand the definition of a binary operation on a set. This leads to an explosion in complexity as one progresses through the subject. Truly mastering a concept involves understanding the full hierarchy of definitions behind it. This can be extremely daunting. If a student looks up a concept in a book they are confronted with new concepts they may not understand. They look these up and are faced with the same problem. Very soon the student has travelled up one branch of the overall hierarchy graph and is essentially clueless about all the others. This is one of the reasons learning mathematics so difficult. The real problem is that the traditional methods of learning are inherently linear so generally fail to give a clear picture of this overall hierarchy.

I am currently in the early stages of developing software to address this shortcoming. More precisely, I am developing a graphical encyclopaedia of Mathematics, which will allow students to navigate around different mathematical concepts whilst having a visual representation of their position in the overall hierarchy. Using this, students will have an easy way to systematically develop their understanding in a way which is intuitive and manageable. In conjunction with the more traditional methods of teaching this will effectively enhance the overall process of learning mathematics.

Teaching Improvement Activities

I am constantly working to improve my teaching and I continue to make positive changes to my courses each time I teach them.

In the summer of 2009 I volunteered to be the supervisor of an IDEAL (The Initiative for Diversity in Education and Leadership) scholar at UC Berkeley. We met twice a week to discuss various mathematical topics, largely focussing on representation theory.

I am currently a member of the Equality and Diversity committee at King's College. We are actively developing methods to promote gender equality throughout the university. This has involved organising open days to promote the scientific contributions of leading female academics. We have also run focus groups with undergraduates to hear their concerns and suggestions.

I am committed to teaching at all levels and recently volunteered to assist King's College London in the development of a new government sponsored mathematics high school. This involved giving several lectures to prospective students, where I introduced them to higher mathematical ideas. This was a very positive experience, and I look forward to doing it again in the future.

Conclusion

Having taught undergraduate and graduate Mathematics for six years I have come to love teaching. I find it stimulating, satisfying and the variety of challenges enhancing. I enjoy the commitment and structure. The exposure to enthusiastic young minds gives my own research context and worth. I bring to teaching mathematical knowledge, true commitment and enthusiasm. My students are part of my mathematical life, and as their teacher it is my responsibility and privilege to be pivotal in their mathematical journey. Teaching is so much more than standing in front of a class explaining something you understand and they do not. It is about the most basic human urge - to connect and share a common passion. Done well, it is of lasting significance for everyone involved.

Appendix A

Math 113

Abstract Algebra

Professor Alexander Paulin

apaulin@math.berkeley.edu
math.berkeley.edu/∼apaulin/
Room 887
Office Hours: Monday 10am -12pm; Wednesday 2pm - 5pm

Course Description: Throughout your mathematical education you've been exposed to many different forms of algebra. The most important include the integers, real and complex numbers, polynomials, functions and matrices. Abstract algebra encompasses all of these and more. Roughly speaking, abstract algebra studies sets equipped with natural laws of composition. The three basic examples we will study are called Groups, Rings, and Fields.
A group is, roughly, a set with a *law of composition* satisfying certain axioms. Examples of groups include the integers equipped with addition, the non-zero real numbers equipped with multiplication, and invertible n by n matrices equipped with matrix multiplication. However, groups arise in many other diverse ways. For example, the symmetries of an object in space naturally comprise a group. The moves that one can do on Rubik's cube comprise a fun example of a group. After studying many examples of groups, we will develop some general theory which concerns the basic principles underlying all groups.
A ring is a set equipped with two laws of composition satisfying certain axioms. An example is the integers with addition and multiplication. Another example is the ring of polynomials. A field is a ring with certain additional nice properties. We will study certain classes of ring which possess many properties in common with the integers.
In addition to the specific topics we will study, which lie at the foundations of much of higher mathematics, an important goal of the course is to expand facility with mathematical reasoning and proofs in general, as a transition to more advanced mathematics courses, and for logical thinking outside of mathematics as well.
Prerequisite(s): Linear Algebra and Differential Equations (Math 54).

Credit: 4

Text(s): We will not be using a specific book for the course. Everything you need to know will be in the notes provided. If you would like to supplement these with further reading I recommend *Abstract Algebra*, by Dummit and Foote, or *Classic Algebra* by P.M. Cohn.

Grade Distribution:

Weekly Homework Assignments	20%
2 Midterm Exams	30%
Final Exam	50%

Tentative Course Outline:

The weekly coverage might change as it depends on the progress of the class.

Week	Content
Week 1	• Introduction; Sets; Functions; Equivalence Relations.
Week 2	• Integers; The Division Algorithm; The Fundamental Theorem of Arithmetic; Congruences.
Week 3	• Groups; Homomorphisms; Lagrange's Theorem.
Week 4	• Cyclic Groups; Permutation Groups and Group Actions; Orbit-Stabiliser.
Week 5	• Dihedral Groups; Normal Subgroups and Isomorphism Theorems; • Review Lecture. • First Midterm Exam.
Week 6	• Direct Products and Direct Sums; Torsion Groups;
Week 7	• Finite Abelian Groups.
Week 8	• Structure Theorem for Finitely Generated Abelian Groups. • Midterm Exam
Week 9	• Rings and Fields; Homomorphisms; Integral Domains;
Week 10	• Field of Fractions; Ideals and Isomorphism Theorems; • Review Lecture. • Second Midterm Exam.
Week 11	• Polynomial Rings; Euclidean Domains;
Week 12	• Unique Factorisation Domains; • Reading assignment: Something interesting
Week 13	• Gauss' Lemma; Eisenstein's Criterion.
Week 14	• Field Extensions and Basic Galois Theory
Week 15	• Review Lecture • Final Exam

Introduction

What is Algebra?

If you ask someone on the street this question the most likely response will be: "Something horrible to do with x, y and z!". If you're lucky enough to bump into a mathematician then you might get something along the lines of: "Algebra is the abstract encapsulation of arithmetic laws of composition".

Algebra is *deep*. In this context, this means that it permeates most of our mathematical intuitions. In fact the first mathematical concepts we ever encounter are the foundation of the subject. Let me summarize the first six years of your mathematical education:

The concept of *Unity*. The number 1.
You probably always understood this, even as a little baby!

$\downarrow$

$\mathbb{N} := \{1, 2, 3...\}$, the natural numbers.
$\mathbb{N}$ comes equipped with two natural operations $+$ and $\times$.

$\downarrow$

$\mathbb{Z} := \{... - 2, -1, 0, 1, 2, ...\}$, the integers.
We form these by using geometric intuition thinking of $\mathbb{N}$ as sitting on a line. $\mathbb{Z}$ also comes with $+$ and $\times$. Addition on $\mathbb{Z}$ has particularly good properties, e.g. additive inverses exist.

$\downarrow$

$\mathbb{Q} := \{\frac{a}{b} \,|a, b \in \mathbb{Z}, b \neq 0\}$, the rational numbers.
We form these by taking $\mathbb{Z}$ and *formally* dividing through by non-negative integers. We can again use geometric insight to picture $\mathbb{Q}$ as points on a line. The rational numbers also come equipped with $+$ and $\times$. This time, multiplication is has particularly good properties, e.g non-zero elements have multiplicative inverses.

Notice that at each stage the operations of $+$ and $\times$ become better behaved. These ideas are very simple, but also profound. We spend years understanding how $+$ and $\times$ behave on $\mathbb{Q}$. e.g.

$$a + b = b + a \quad \forall a, b \in \mathbb{Q},$$

or

$$a \times (b + c) = a \times b + a \times c \quad \forall a, b, c \in \mathbb{Q}.$$

The central idea behind abstract algebra is to define a larger class of objects (sets with extra structure), of which $\mathbb{Z}$ and $\mathbb{Q}$ are definitive members.

$$\begin{aligned}
(\mathbb{Z}, +) &\longrightarrow \textit{Groups} \\
(\mathbb{Z}, +, \times) &\longrightarrow \textit{Rings} \\
(\mathbb{Q}, +, \times) &\longrightarrow \textit{Fields}
\end{aligned}$$

In linear algebra the analogous idea is

$$(\mathbb{R}^n, +, \text{multiplication by scalars}) \longrightarrow Vector\ Spaces\ over\ \mathbb{R}$$

The amazing thing is that these vague ideas mean something very precise and have far far more depth than one could ever imagine.

Appendix B

Groups

Definition. *Let G be a set. A* ***binary operation*** *is a map of sets:*

$$* : G \times G \to G.$$

For ease of notation we write $*(a, b) = a * b \ \forall a, b \in G$. Any binary operation on G gives a way of *combining* elements and we say that $a * b$ is the composition of a and b. As we have seen, if $G = \mathbb{Z}$ then $+$ and $\times$ are natural examples of binary operations.

Definition. *A* ***group*** *is a set G, together with a binary operation $*$, such that the following hold:*

1. *(Associativity): $(a * b) * c = a * (b * c) \ \forall a, b, c \in G$.*

2. *(Existence of identity): $\exists e \in G$ such that $a * e = e * a = a \ \forall a \in G$.*

3. *(Existence of inverses): Given $a \in G, \exists b \in G$ such that $a * b = b * a = e$.*

Remarks. 1. *We have seen five different examples thus far: $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{Q}\backslash\{0\}, \times)$, $(\mathbb{Z}/m\mathbb{Z}, +)$, and $(\mathbb{Z}/m\mathbb{Z}, \times)$. Another example is that of a real vector space under addition. Note that $(\mathbb{Z}, \times)$ is* ***not*** *a group. Also note that this gives examples of groups which are both finite and infinite. The more mathematics you learn the more you'll see that groups are everywhere.*

2. *A set with a binary operation is called a* ***monoid*** *if we demand that only the first two properties hold. From this point of view, a group is a monoid in which every element is invertible. $(\mathbb{Z}, \times)$ is a monoid.*

3. *Observe that in all of the examples I've given the binary operation is commutative, i.e. $a * b = b * a \ \forall a, b \in G$. We do not include this in our definition as this would be too restrictive as we'll see later. For example the set of invertible $n \times n$ matrices with real coefficients forms a group under matrix multiplication. However we know that matrix multiplication does not commute in general.*

So a group is a set with extra structure. In set theory we have the natural concept of a map between sets. The following is the analogous concept for groups:

Definition. *Let $(G, *)$ and $(H, \circ)$ be two groups. A* ***homomorphism*** *f, from G to H is a map of sets $f : G \to H$, such that $f(x * y) = f(x) \circ f(y) \ \forall x, y \in G$. If $G = H$ and $f = Id_G$ we call f the identity homomorphism.*

Remarks. 1. *Intuitively one should thing about a homomorphism as a map of sets which preserves the underlying binary operations. It's the same idea as a linear map between vector spaces.*

2. *A homomorphism $f : G \to H$ which is bijective is called an* ***isomorphism****. Two groups are said to be* ***isomorphic*** *if there exists an isomorphism between them. Intuitively two groups being isomorphic means that they are the "same" group, but viewed from different perspectives.*

3. *A homomorphism from a group to itself (i.e. $f : G \to G$) is called an* ***endomorphism****. An endomorphism which is also an isomorphism is called an* ***automorphism****.*

Proposition. *Let $(G, *), (H, \circ)$ and $(M, \Box)$ be three groups. Let $f : G \to H$ and $g : H \to M$ be homomorphism. Then the composition $gf : G \to M$ is a homomorphism.*

Proof. Let $x, y \in G$. $gf(x * y) = g(f(x) \circ f(y)) = gf(x) \Box gf(y)$. $\Box$

Remark. *Composition of homomorphism gives the collection of endomorphisms of a group the structure of a monoid. The subset of automorphisms has the stucture of a group under composition. We denote it by $Aut(G)$.*

Proposition. *Let $(G, *)$ be a group. The identity element is unique.*

Proof. Assume $e, e' \in G$ both behave like the identity. Then $e = e * e' = e'$. $\Box$

Proposition. *Let $(G, *)$ be a group. For $a \in G$ there is only one element which behaves like the inverse of a.*

Proof. Assume $a \in G$ has 2 inverses, $b, c \in G$. Then:

$$\begin{aligned} (a * b) &= e \\ \Rightarrow c * (a * b) &= c * e \\ \Rightarrow (c * a) * b &= c \quad \text{(associativity and identity)} \\ \Rightarrow \qquad e * b &= c \\ \Rightarrow \qquad\quad b &= c \end{aligned}$$

$\Box$

The first proposition tells us that we can write $e \in G$ for the identity and it is well defined. Similarly the second proposition tells us that for $a \in G$ we can write $a^{-1} \in G$ for the inverse in a well defined way. The proof of the second result gives a good example of how we prove results for abstract groups. We can only use the axioms, nothing else.

We also have an analogue of the cancellation law:

Proposition. *Let $a, b, c \in G$ a group. Then*

$$a * c = a * b \Rightarrow c = b \text{ and } c * a = b * a \Rightarrow c = b$$

Proof. Compose on left or right by $a^{-1} \in G$, then apply the associativity and inverses and identity axioms. $\Box$

Definition. *A group* $(G, *)$ *is called* ***Abelian*** *if it also satisfies*

$$a * b = b * a \; \forall a, b \in G.$$

This is also called the commutative property.

In linear algebra, we can talk about subspaces of vector spaces. We have an analogous concept in group theory.

Definition. *Let* $(G, *)$ *be a group. A* ***subgroup*** *of* G *is a subset* $H \subset G$ *such that*

1. $e \in H$
2. $x, y \in H \Rightarrow x * y \in H$
3. $x \in H \Rightarrow x^{-1} \in H$

Remarks.

1. *A subgroup is naturally a group under the induced binary operation. It clearly has the same identity element.*
2. *If* $m \in \mathbb{N}$*, then the subset* $m\mathbb{Z} := \{ma | a \in \mathbb{Z}\}$*is a subgroup of* $(\mathbb{Z}, +)$.

Proposition. *If* $H, K \subset G$ *are subgroups* $\Rightarrow H \cap K \subset G$ *is a subgroup.*

Proof.

1. As H, K subgroups, $e \in H$and$e \in K \Rightarrow e \in H \cap K$.
2. Let $x, y \in H \cap K \Rightarrow x * y \in H$ and $x * y \in K \Rightarrow x * y \in H \cap K$.
3. Let $x \in H \cap K \Rightarrow x^{-1} \in H$ and $x^{-1} \in K \Rightarrow x^{-1} \in H \cap K$.

$\square$

This result clearly extends to any collection of subgroups of G.

Appendix C

CM222, Linear Algebra **Exercise Sheet 3**

Due in tutorials, 24 Oct.

1. Which of the following formulas define linear maps $L : \mathcal{P} \to \mathcal{P}$?
   (a) $(L(p))(x) = x^2 p(x)$;
   (b) $(L(p))(x) = x^2 + p(x)$;
   (c) $(L(p))(x) = (p(x))^2$;
   (d) $(L(p))(x) = p(x^2)$;
   (e) $(L(p))(x) = p''(x)$;
   (f) $(L(p))(x) = p(p(x))$;
   (g) $(L(p))(x) = x^2 p''(2x + 2) + 2p(x^2)$.

2. Suppose that U, V and W are vector spaces, and that $f : V \to W$ and $g : U \to V$ are linear maps.
   (a) Prove that the composite $f \circ g$ is a linear map from U to W.
   (b) Prove that if f is an isomorphism then so is its inverse map from W to V.

3. Let V and W be vector spaces, S a subset of V, and $f : V \to W$ a linear map.
   (a) Prove that $\ker f$ is a subspace of V.
   (b) Prove that the image of f is a subspace of W.
   (c) Prove that f is injective if and only if $\ker f = \{0\}$.

4. Let $f : M_{2,2}(\mathbb{R}) \to M_{2,2}(\mathbb{R})$ be the map defined by $f(A) = A - A^t$ (where A^t denotes the transpose of A).
   (a) Prove that f is a linear map.
   (b) Find a basis for $\ker(f)$.
   (c) Find the rank and nullity of f.
   (d) Find a basis for $\mathrm{im}(f)$.

5. Find the rank and nullity of each of the following matrices:
   (a) $\begin{pmatrix} 2 & -1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$.
   (b) $\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$.
   (c) $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map defined by reflection in the line $x + y = 0$.

   (a) Find the matrix A_f of f with respect to the standard basis S.

   (b) Find the matrix A'_f of f with respect to the basis

$$S' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

   (c) Verify that $A'_f = P A_f P^{-1}$ where P is the transition matrix from S to S'.

7. Let $\mathcal{P}_n$ denote the vector space of real polynomials of degree at most n. Find the matrices of the following linear maps with respect to the bases $\{1, x, x^2, x^3\}$ for $\mathcal{P}_3$ and $\{1, x, x^2, x^3, x^4\}$ for $\mathcal{P}_4$.

   (a) $\phi : \mathcal{P}_3 \to \mathcal{P}_4$ defined by $(\phi(p))(x) = xp(x)$;

   (b) $\psi : \mathcal{P}_4 \to \mathcal{P}_4$ defined by $(\psi(p))(x) = p(-x)$;

   (c) $\xi : \mathcal{P}_4 \to \mathcal{P}_3$ defined by $(\xi(p))(x) = p'(x)$ (the derivative of $p(x)$);

   (d) $\mu : \mathcal{P}_3 \to \mathcal{P}_3$ defined by $\mu = \xi \circ \psi \circ \phi$.

1. Which of the following formulas define linear maps $L : \mathcal{P} \to \mathcal{P}$?

   (a) $(L(p))(x) = x^2 p(x)$;
   Solution: YES. Suppose $p, q \in \mathcal{P}$. Then $L(p)(x) = x^2 p(x)$, $L(q)(x) = x^2 q(x)$ and $(p+q)(x) = p(x) + q(x)$, so

$$\begin{aligned} L(p+q)(x) &= x^2((p+q)(x)) \\ &= x^2(p(x)+q(x)) \\ &= x^2 p(x) + x^2 q(x) \\ &= L(p)(x) + L(q)(x) \\ &= (L(p)+L(q))(x). \end{aligned}$$

   Therefore $L(p+q) = L(p) + L(q)$.
   If $p \in \mathcal{P}$ and $\alpha \in \mathbb{R}$, then

$$L(\alpha p)(x) = x^2(\alpha p(x)) = \alpha(x^2 p(x)) = \alpha L(p)(x),$$

   so $L(\alpha p) = \alpha L(p)$.

   (b) $(L(p))(x) = x^2 + p(x)$;
   Solution: NO. Note for example that $L(0) = x^2 \neq 0$, so L cannot be a linear map.

   (c) $(L(p))(x) = (p(x))^2$;
   Solution: NO. Note for example that for any $p \in \mathcal{P}$, $\alpha \in \mathbb{R}$, we have

$$L(\alpha p)(x) = (\alpha p(x))^2 = \alpha^2 (p(x))^2 = \alpha^2 L(p)(x).$$

   Taking any α so that $\alpha \neq \alpha^2$ (e.g., $\alpha = 2$) and any p so that $L(p) \neq 0$ (e.g., $p(x) = 1$, the constant function), we get

$$L(\alpha p) = \alpha^2 L(p) \neq \alpha L(p).$$

   (d) $(L(p))(x) = p(x^2)$;
   Solution: YES. If $p, q \in \mathcal{P}$, then

$$L(p+q)(x) = (p+q)(x^2) = p(x^2) + q(x^2) = L(p)(x) + L(q)(x),$$

   so $L(p+q) = L(p) + L(q)$.
   If $\alpha \in \mathbb{R}$ and $p \in \mathcal{P}$, then

$$L(\alpha p)(x) = \alpha p(x^2) = \alpha L(p)(x),$$

   so $L(\alpha p) = \alpha L(p)$.

   (e) $(L(p))(x) = p''(x)$;
   Solution: YES. If $p, q \in \mathcal{P}$, then

$$L(p+q)(x) = (p+q)''(x) = p''(x) + q''(x) = L(p)(x) + L(q)(x),$$

   and if $\alpha \in \mathbb{R}$, then

$$L(\alpha p)(x) = \alpha p''(x) = \alpha L(p)(x),$$

   so $L(\alpha p) = \alpha L(p)$.

   (f) $(L(p))(x) = p(p(x))$;
   Solution: NO. Take for example $p(x) = x$ and $\alpha = 2$. Then $L(p)(x) = p(x) = x$, so $\alpha L(p)(x) = 2$. To compute $L(\alpha p)$, let $q = \alpha p$ so $q(x) = 2x$ and

$$L(q)(x) = q(q(x)) = q(2x) = 4x,$$

   so $\alpha L(p) \neq L(\alpha p)$.

(g) $(L(p))(x) = x^2p''(2x+2) + 2p(x^2)$.
Solution: YES. If $p, q \in \mathcal{P}$, then

$$\begin{aligned} L(p+q)(x) &= x^2(p+q)''(2x+2) + 2(p+q)(x^2) \\ &= x^2(p''+q'')(2x+2) + 2(p(x^2)+q(x^2)) \\ &= x^2p''(2x+2) + x^2q''(2x+2) + 2p(x^2) + 2q(x^2) \\ &= (x^2p''(2x+2) + 2p(x^2)) + (x^2q''(2x+2) + 2q(x^2)) \\ &= L(p(x)) + L(q(x)). \end{aligned}$$

Similarly if $p \in \mathcal{P}$, $\alpha \in \mathbb{R}$, then

$$\begin{aligned} L(\alpha p)(x) &= x^2(\alpha p)''(2x+2) + 2(\alpha p)(x^2) \\ &= \alpha x^2 p''(2x+2) + \alpha(2p(x^2)) \\ &= \alpha L(p)(x). \end{aligned}$$

2. Suppose that U, V and W are vector spaces, and that $f : V \to W$ and $g : U \to V$ are linear maps.

(a) Prove that the composite $f \circ g$ is a linear map from U to W.
Solution: If $u, u' \in U$, then $(f(g(u+u')) = f(g(u)+g(u'))$ (since g is linear), and this is equal to $f(g(u)) + f(g(u'))$ (since f is linear). Therefore $(f \circ g)(u+u') = (f \circ g)(u) + (f \circ g)(u')$.
Similarly, if $\alpha \in \mathbb{R}$ and $u \in U$, then $f(g(\alpha u)) = f(\alpha g(u)) = \alpha f(g(u))$, so $(f \circ g)(\alpha u) = \alpha(f \circ g)(u)$.

(b) Prove that if f is an isomorphism then so is its inverse from W to V.
Solution: Let g be the inverse of f, so $f(g(w)) = w$ for all $w \in W$ and $g(f(v))$ for all $v \in V$. We know that g is bijective; we must show that $g(w+w') = g(w)+g(w')$ and $g(\alpha w) = \alpha g(w)$ for all $w, w' \in W$, $\alpha \in \mathbb{R}$. Since f is injective ($f(v_1) = f(v_2) \Rightarrow v_1 = v_2$), the first equation follows if we show that $f(g(w+w')) = f(g(w)+g(w'))$. Since f is linear,

$$f(g(w)+g(w')) = f(g(w)) + f(g(w')) = w + w' = f(g(w+w')).$$

Similarly $f(\alpha g(w)) = \alpha f(g(w)) = \alpha w = f(g(\alpha w)$ implies that $g(\alpha w) = \alpha g(w)$.

3. Let V and W be vector spaces and $f : V \to W$ a linear map.

(a) Prove that $\ker f$ is a subspace of V.
Solution: If $v, v' \in \ker f$, then $f(v) = f(v') = 0$, so $f(v+v') = f(v) + f(v') = 0 + 0 = 0$ and $v + v' \in \ker f$. If $\alpha \in \mathbb{R}$ (and $v \in \ker f$), then $f(\alpha v) = \alpha f(v) = \alpha.0 = 0$, so $\alpha v \in \ker f$.

(b) Prove that the image of f is a subspace of W.
Solution: If $w, w' \in \operatorname{im} f$, then $w = f(v)$ and $w' = f(v')$ for some $v, v' \in V$. Therefore $w + w' = f(v) + f(v') = f(v+v') \in \operatorname{im} f$. If $\alpha \in \mathbb{R}$, then $\alpha w = \alpha f(v) = f(\alpha v) \in \operatorname{im} f$.

(c) Prove that f is injective if and only if $\ker f = \{0\}$.
Solution: Suppose that f is injective. If $v \in \ker f$, then $f(v) = 0 = f(0)$ implies that $v = 0$ (since f is injective). So $\ker f = \{0\}$.
Suppose that $\ker f = 0$. If $f(v) = f(v')$, then $f(v - v') = f(v) - f(v') = 0$, so $v - v' \in \ker f$. It follows that $v - v' = 0$, so $v = v'$. Therefore f is injective.

4. Let $f : M_{2,2}(\mathbb{R}) \to M_{2,2}(\mathbb{R})$ be the map defined by $f(A) = A - A^t$ (where A^t denotes the transpose of A).

(a) Prove that f is a linear map.

Solution: Note that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2,2}(\mathbb{R})$, then

$$f(A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix}.$$

So if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ and $\alpha\alpha' \in \mathbb{R}$, then

$$\begin{aligned} f(A + A') &= \begin{pmatrix} 0 & (b+b) - (c+c') \\ (c+c') - (b+b') & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix} + \begin{pmatrix} 0 & b'-c' \\ c'-b' & 0 \end{pmatrix} \\ &= f(A) + f(A'). \end{aligned}$$

(b) Find a basis for $\ker(f)$.
Solution: From the first formula in the solution to (a), we see that $f(A) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ means that $b = c$. So A is in $\ker(f)$ if and only if A has the form $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$, which means that

$$A = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore A is in the span of the set of matrices $\{A_1, A_2, A_3\}$ where

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

These matrices are linear independent since

$$\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{pmatrix}$$

is the zero matrix if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore $\{A_1, A_2, A_3\}$ is a basis for $\ker(f)$.

(c) Find the rank and nullity of f.
Solution: From part (b) we see that $\text{nullity}(f) = 3$. Since $M_{2,2}(\mathbb{R})$ has dimension 4, the rank-nullity theorem shows that $\text{rank}(f) = 1$.

(d) Find a basis for $\text{im}(f)$.
Solution: By part (c), the image of has dimension 1, so it suffices to find a non-zero matrix B in $\text{im}(f)$. Taking for example

$$B = f\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we have $\{B\}$ as a basis for $\text{im}(f)$.

5. Find the rank and nullity of each of the following matrices:

(a) $\begin{pmatrix} 2 & -1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$.
Solution: Apply row operations, or note that the columns are linearly independent, so the rank is 2 and the nullity is $2 - 2 = 0$.

(b) $\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$.
Solution: Apply row operations, or note that the rows are linearly independent, so the rank is 2 and the nullity is $3 - 2 = 1$.

(c) $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$.

Solution: Applying row operations gives $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, so the rank is 2 and the nullity is $3 - 2 = 1$.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map defined by reflection in the line $x + y = 0$.

   (a) Find the matrix A_f of f with respect to the standard basis S.
   Solution: Since $f((1,0)^t) = (0,-1)^t$ and $f((0,1)^t) = (-1,0)^t$, the matrix is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

   (b) Find the matrix A'_f of f with respect to the basis

   $$S' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

   Solution: Since $f((1,1)^t) = -(1,1)^t$ and $f((1,-1)^t) = (1,-1)^t$, the matrix is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

   (c) Verify that $A'_f = PA_fP^{-1}$ where P is the transition matrix from S to S'. **Solution:** The transition matrix from S' to S is $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, so this is P^{-1} and $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Therefore

   $$\begin{aligned} PA_fP^{-1} &= \tfrac{1}{2}\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = A'_f. \end{aligned}$$

7. Let $\mathcal{P}_n$ denote the vector space of real polynomials of degree at most n. Find the matrices of the following linear maps with respect to the bases $\{1, x, x^2, x^3\}$ for $\mathcal{P}_3$ and $\{1, x, x^2, x^3, x^4\}$ for $\mathcal{P}_4$.

   (a) $\phi : \mathcal{P}_3 \to \mathcal{P}_4$ defined by $(\phi(p))(x) = xp(x)$;
   Solution: Applying ϕ to the given basis vectors for $\mathcal{P}_3$, we have $\phi(1) = x$, $\phi(x) = x^2$, $\phi(x^2) = x^3$ and $\phi(x^3) = x^4$. These have coordinate vectors $(0,1,0,0,0)^t$, $(0,0,1,0,0)^t$, $(0,0,0,1,0)^t$ and $(0,0,0,0,1)^t$ with respect to the given basis for $\mathcal{P}_4$. Using these as the columns give the matrix

   $$A_\phi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   (b) $\psi : \mathcal{P}_4 \to \mathcal{P}_4$ defined by $(\psi(p))(x) = p(-x)$;
   Solution: Applying ψ to the given basis vectors for $\mathcal{P}_4$, we get $\psi(1) = 1$, $\psi(x) = -x$, $\psi(x^2) = x^2$, $\psi(x^3) = -x^3$ and $\psi(x^4) = x^4$, so the matrix is

   $$A_\phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

   (c) $\xi : \mathcal{P}_4 \to \mathcal{P}_3$ defined by $(\xi(p))(x) = p'(x)$ (the derivative of $p(x)$);
   Solution: $\xi(1) = 0$, $\xi(x) = 1$, $\xi(x^2) = 2x$, $\xi(x^3) = 3x^2$ and $\xi(x^4) = 4x^3$, so

   $$A_\xi = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

(d) $\mu : \mathcal{P}_3 \to \mathcal{P}_3$ defined by $\mu = \xi \circ \psi \circ \phi$.
Solution: $A_\mu = A_\xi A_\psi A_\phi =$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Appendix D

Full Name: ____________________
(block capitals)

Student Number: ____________________

Tutorial Group: __________

5CCM222A Linear Algebra: Class Test 2

CALCULATORS MAY NOT BE USED

ANSWER GRID: To earn full marks, you must correctly answer YES or NO in EVERY BOX. Points will be deducted for each box that is filled in incorrectly. You may leave boxes blank, in which case no marks will be given or deducted for those boxes. If you change an answer, make sure that your final answer is clearly indicated.

	a	b	c	d
1				
2				
3				
4				
5				

TIME: 30 minutes

MARKS: Each correct box = +1, incorrect = −1.

1. Suppose that $S = \{v_1, v_2, v_3\}$ is a basis for a three-dimensional real vector space V, and $f : V \to \mathbb{R}^4$ is a linear map whose matrix is

$$\begin{pmatrix} 1 & 2 & -2 \\ -1 & 0 & 2 \\ 2 & 1 & 1 \\ 0 & 3 & 0 \end{pmatrix}$$

with respect to the basis S of V and the standard basis of $\mathbb{R}^4$. Which of the following appear as coordinates in the vector $f(v_1 + 2v_3)$?

(a) 0
(b) 1
(c) 2
(d) 3

2. Let $g : U \to V$ and $f : V \to W$ be linear maps of finite-dimensional real vector spaces. Suppose that $f \circ g$ is injective. Which of the following statements must then be true?

(a) $\dim(U) \leq \dim(V)$
(b) f is injective
(c) g is injective
(d) $\dim(U) \leq \dim(W)$

3. For which of the following values of μ does the non-homogeneous linear system:

$$\begin{array}{rcrcr} x & + & \mu y & = & 2 \\ x & + & (\mu+1)y & = & \mu+2 \\ 2x & + & \mu y & = & 3 \end{array}$$

have at least one solution?

(a) 2
(b) -1
(c) 1
(d) 0

4. Which of the following matrices have determinant 2?

(a) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

5. Let A be a real $n \times n$-matrix. Which of the following statements are equivalent to A having non-zero determinant?

(a) $AB = I$ for some real $n \times n$-matrix B.
(b) The equation $A\mathbf{x} = \mathbf{0}$ has no non-zero solutions $\mathbf{x} \in \mathbb{R}^n$.
(c) For every vector $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} \in \mathbb{R}^n$.
(d) The rows of A are linearly dependent.

End of Test

King's College London

UNIVERSITY OF LONDON

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the authority of the Academic Board.

ATTACH THIS PAPER TO YOUR SCRIPT **USING THE STRING PROVIDED**

Candidate No: .... **Desk No:** ....

BSC AND MSCI EXAMINATION

CM222 LINEAR ALGEBRA

SUMMER 2013

TIME ALLOWED: THREE HOURS

THIS PAPER CONSISTS OF TWO SECTIONS, SECTION A AND SECTION B.

SECTION A CONTRIBUTES HALF THE TOTAL MARKS FOR THE PAPER.

ANSWER ALL QUESTIONS IN SECTION A.

ALL QUESTIONS IN SECTION B CARRY EQUAL MARKS, BUT IF MORE THAN TWO ARE ATTEMPTED, THEN ONLY THE BEST TWO WILL COUNT.

NO CALCULATORS ARE PERMITTED.

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

TURN OVER WHEN INSTRUCTED

SECTION A

A 1. Throughout this question let V be a vector space over $\mathbb{R}$.

(a) Let $S = \{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\}$ be a subset of V. Define each of the following properties:

(i) S is linearly independent.

(ii) S is a spanning set for V.

(iii) S is a basis for V.

(iv) V is finite dimensional.

(b) If S' is a finite subset of Span(S), the spanning set of S, prove that Span$(S') \subset$ Span(S).

(c) Let $W \subset V$ be a subset. Define what it means for W to be a subspace of V. Give an example of a non-empty subset of $\mathbb{R}^3$ which is closed under addition but **not** under scalar multiplication.

(d) Let $V = \mathbb{R}^4$ and consider the four vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ and $\boldsymbol{v}_4$, each belonging to V, where,

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 2 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 6 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 4 \end{pmatrix}, \boldsymbol{v}_4 = \begin{pmatrix} 5 \\ 0 \\ 3 \\ 10 \end{pmatrix}.$$

(i) Find a linear dependency between $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ and $\boldsymbol{v}_4$.

(ii) Let $W = \text{Span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4)$. Find the dimension of W and give a basis for W. Justify your answer.

A 2. Let V and W be complex vector spaces.

(a) Define what it means for a map $f : V \to W$ to be linear. Prove that if $\mathbf{0}_V, \mathbf{0}_W$ are the zero vectors in V and W respectively then

$$f \text{ linear } \Rightarrow f(\mathbf{0}_V) = \mathbf{0}_W$$

(b) Let $f : V \to W$ be a linear map. Define each of the following:

(i) the kernel of f.

(ii) the image of f.

(c) Prove that if $f : V \to W$ is linear then $im(f) \subset W$ is a subspace. Further prove that if V is finite dimensional then $im(f)$ is finite dimensional.

(d) Fix the basis $\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\} \subset \mathbb{C}^2$ (you do not need to show this is a basis). Let f be the linear map from $\mathbb{C}^2$ to $\mathbb{C}^3$ such that

$$f\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3\\1\\2\end{pmatrix}, f\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}.$$

Write down the matrix for f with respect to the basis $\{\begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\} \subset \mathbb{C}^2$ and the standard basis for $\mathbb{C}^3$. What is the nullity of f? What is the rank of f?

A 3. (a) Let $\mathbf{A}$ be an $n \times n$ matrix with complex coefficients. Define each of the following:

(i) $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathbf{A}$.

(ii) $\boldsymbol{v} \in \mathbb{C}^n$ is an eigenvector of $\mathbf{A}$.

(iii) $\mathbf{A}$ is diagonalisable.

.

(b) Let $\mathbf{A}$ be the matrix $\begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & 1 \\ -1 & 1 & 4 \end{pmatrix}$.

(i) Find all eigenvalues of $\mathbf{A}$.

(ii) Write down an orthogonal real matrix $\mathbf{P}$ such that $\mathbf{P}^T\mathbf{A}\mathbf{P}$ is a diagonal matrix.

A 4. (a) Let V be a complex vector space. Give the definition of a complex inner product on V.

(b) Prove that the following is a real inner product on $\mathbb{R}^3$:

$$\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \rangle = 2xx' + 3yy' + 4zz'.$$

(c) Prove that the following is **not** a real inner product on $\mathbb{R}^3$:

$$\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \rangle = x^2x' + y^2y' + z^2z'.$$

(d) Let $\mathbb{C}^3$ be equipped with the standard complex inner product. Apply the Gram-Schmidt process to the basis $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} -1 \\ i \\ 1 \end{pmatrix}$ and $\boldsymbol{v}_3 = \begin{pmatrix} 0 \\ -1 \\ i+1 \end{pmatrix}$ in $\mathbb{C}^3$. Using this write down a linear map $f : \mathbb{C}^3 \to \mathbb{C}^3$ with $im(f) = Span \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$ and such that given any $\boldsymbol{v} \in ker(f)$ and $\boldsymbol{w} \in im(f)$ we have $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0$.

SECTION B

B 5. Let V be a finite dimensional complex vector space with ordered basis $S = \{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\}$.

(a) For $\boldsymbol{v} \in V$ let $\boldsymbol{v}_S \in \mathbb{C}^n$, be $\boldsymbol{v}$ written in coordinates with respect to S. Prove that this defines a linear map

$$\phi_S : V \to \mathbb{C}^n$$

$$\boldsymbol{v} \mapsto \boldsymbol{v}_S.$$

Further prove that ϕ_S is an isomorphism.

(b) Let W be a second complex vector space and assume $f : V \to W$ is an isomorphism. Prove that the inverse map $f^{-1} : W \to V$ is an isomorphism. Using this, or otherwise, prove that if two vector spaces have the same dimension then they are isomorphic.

(c) Let $f : V \to \mathbb{C}^n$ be an isomorphism. Using part (b) find an ordered basis T for V such that $f = \phi_T$. Carefully justify your answer. (Hint: consider what maps to the standard basis)

(d) Let $\mathcal{P}_n(X)$ denote the complex vector space of polynomials in the variable X with complex complex coefficents of degree less than or equal to $n-1$. Let φ be the linear map

$$\varphi : \mathcal{P}(X) \to \mathcal{P}(X)$$

$$f(X) \to \frac{df(X)}{dX}.$$

Determine the nullity and rank of φ. If you use the rank-nullity theorem state it clearly. Find two ordered bases $S, T \subset \mathcal{P}(X)$ such that the matrix associated to φ with respect to S and T is of the form:

$$\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where $r = rank(\varphi)$ and $\mathbf{I}_r$ is the $r \times r$ identity matrix.

Is φ diagonalisable? Carefully justify your answer.

B 6. Let $\mathbf{A}, \mathbf{B}$ be two $n \times n$ matrices with complex entries.

(a) Define the characteristic polynomial, $p_{\mathbf{A}}(X)$, of $\mathbf{A}$. Prove that $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathbf{A}$ if and only if $p_{\mathbf{A}}(\lambda) = 0$. You may use any results about invertible matrices as long as you state them clearly.

(b) Define what it means for $\mathbf{A}$ to be similar to $\mathbf{B}$. Prove that if $\mathbf{A}$ and $\mathbf{B}$ are similar then $p_{\mathbf{A}}(X) = p_{\mathbf{B}}(X)$. You may use any results about determinants as long as you state them clearly.

(c) Define the minimal polynomial $m_{\mathbf{A}}(X)$ of $\mathbf{A}$. Prove that if $\mathbf{A}$ and $\mathbf{B}$ are similar then $m_{\mathbf{A}}(X) = m_{\mathbf{B}}(X)$.

(d) Give examples of matrices $\mathbf{A}$ and $\mathbf{B}$ such that

(i) $p_{\mathbf{A}}(X) = p_{\mathbf{B}}(X)$ but $m_{\mathbf{A}}(X) \neq m_{\mathbf{B}}(X)$.

(ii) $m_{\mathbf{A}}(X) = m_{\mathbf{B}}(X)$ but $p_{\mathbf{A}}(X) \neq p_{\mathbf{B}}(X)$.

(e) Let $\mathbf{T}$ be an upper triangular $n \times n$ complex matrix. Prove that if $p_{\mathbf{T}}(X) = a_0 + a_1 X + \cdots + a_{n-1} X^{n-1} + X^n$ then $a_{n-1} = -tr(\mathbf{T})$. Using this prove that the same result holds for any complex $n \times n$ matrix $\mathbf{A}$. You may use any results from lectures as long as they are clearly stated.

B 7. Let V be a complex inner product space, with inner product $\langle\ ,\rangle$.

(a) Define what it means for a subset $S = \{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\} \subset V$ to be orthonormal. Prove that if S is orthonormal then it is linearly independent.

(b) Define what it means for an $n \times n$ complex matrix $\mathbf{U}$ to be unitary. Prove that $\mathbf{U}$ is unitary if and only if its columns form an orthonormal basis for $\mathbb{C}^n$ with respect to the standard inner product.

(c) State and prove the Pythagorean theorem for complex inner product spaces. Using this, state and prove the Cauchy-Schwartz inequality for complex inner product spaces. Using this, state and prove the triangle inequality for complex inner product spaces.

B 8. Let $\mathbf{A}$ and $\mathbf{B}$ be complex $n \times n$ matrices.

(a) Define the adjoint of $\mathbf{A}$, denoted $\mathbf{A}^*$. Define what it means for $\mathbf{A}$ to be self adjoint. Define what it means for $\mathbf{A}$ to be normal. Define what it means for $\mathbf{A}$ to be unitarily diagonalisable.

(b) Prove that for all $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{C}^n$ we have

$$\langle \mathbf{A}\boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{v}, \mathbf{A}^* \boldsymbol{w} \rangle.$$

Using this prove that all the eigenvalues of a self adjoint matrix are real.

(c) Assume now that $\mathbf{A}$ is a real $n \times n$ matrix. Prove that if $\mathbf{A}$ is orthogonally diagonalisable then it is symmetric.

(d) Prove that if $\mathbf{A}$ is orthogonally similar to an upper triangular matrix $\mathbf{T}$ which is **not** diagonal then it is **not** orthogonally diagonalisable.

(e) Is it possible that a (not orthogonally) diagonalisable matrix $\mathbf{A}$, can satisfy the same condition as in (d)? Justify your answer, carefully stating any results you use.

Final Page