Systems of Linear Equations and Row Reduction

Linear Algebra $\simeq$ Mathematics which emerges from trying to solve linear systems.

Q/: What are linear systems and how do we solve them?

Examples

1/ Find all x and y such that both $x-y=-1$ and $4x+2y=8$.

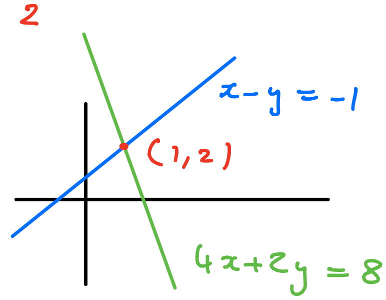
Method 1: Solve one equation in y & substitute into second equation.

Problem: Easy in this case but with more variables/equations it would be confusing.

Method 2: Cleverly combine equations to simplify.

$$\begin{aligned} x-y&=-1\\ 4x+2y&=8 \end{aligned} \xrightarrow{\text{Take } 4\times(\text{Equation } 1) \text{ from Equation } 2} \begin{aligned} x-y&=-1\\ 0\cdot x+6y&=12 \end{aligned} \xrightarrow{\text{Solve 2nd in } y. \text{ Then solve 1st in } x} \begin{aligned} y&=2\\ x&=1 \end{aligned}$$

Geometric Picture:



Other Possibilities: Lines don't intersect (No solutions)

Lines are the same (infinitely many solutions)

2/ Find x, y, z such that $x+3y+5z=1$ and $x+y+7z=2$.

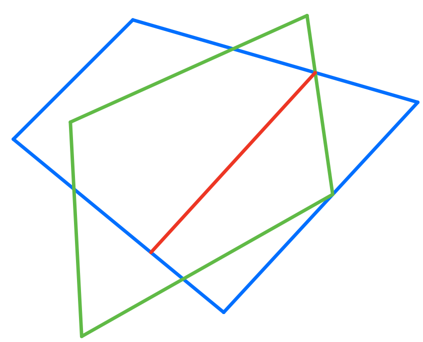
$$\begin{aligned} x+3y+5z &= 1 \\ x+y+7z &= 2 \end{aligned} \xrightarrow[\text{Subtract 1st from 2nd}]{} \begin{aligned} x+3y+5z &= 1 \\ -2y+2z &= 1 \end{aligned}$$

$$-2y+2z=1 \Rightarrow y = z - \tfrac{1}{2}$$

$$x+3y+5z=1 \Rightarrow x = 1-3\left(z-\tfrac{1}{2}\right)-5z = \tfrac{5}{2}-8z$$

$\Rightarrow$ General solution is $\left(\frac{5}{2}-8z,\ z-\frac{1}{2},\ z\right)$ where z free.

Geometric Picture : Two planes intersecting in a line



General Situation :

Linear Equation in n-variable : $a_1 x_1 + \dots + a_n x_n = b$

constant (coefficient) — variable — constant

Linear System :
$$\begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

Example 1 above : $n = 2$, $m = 2$

Example 2 above : $n = 3$, $m = 2$

Called "augmented" matrix of linear system

Matrix Notation :
$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right)$$

Fancy word for rectangular array of numbers

Coefficient Matrix denoted A.

Aim : Find general solution to linear system.

Strategy : Cleverly combine equations to eliminate variables.

Row Operations on Linear System / augmented matrix :

1/ Scale equation / row by non-zero number

2/ Add / subtract one equation / row to another

3/ Rearrange equations / rows.

Q/ : What form should our simplified augmented matrix be ?

Previous Examples :

Less variables as we go down. Can solve by backwards substitution

1/ $\left(\begin{array}{cc|c} 1 & -1 & -1 \\ 4 & 2 & 8 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 6 & 12 \end{array}\right)$

2/ $\left(\begin{array}{ccc|c} 1 & 3 & 5 & 1 \\ 1 & 1 & 7 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 5 & 1 \\ 0 & -2 & 2 & 1 \end{array}\right)$

Definition A matrix is in <u>echelon form</u> if

1/ All non-zero rows are above any zero rows.

2/ Each non-zero leading entry of a row is to the <u>left</u> of any non-zero leading entries of lower rows.

Examples

$$\begin{pmatrix} \blacksquare & * & * \\ 0 & 0 & \blacksquare \end{pmatrix}, \quad \begin{pmatrix} 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A matrix is in <u>reduced echelon form</u> if, in addition,

3/ Non-zero leading entries of rows are 1

4/ There are zeros <u>above</u> a leading non-zero entry.

Examples

$$\begin{pmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & * & 0 & * & 0 \\ 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Fact: Using row operations a matrix may be put into a <u>unique</u> reduced echelon form.

Key Observation: Row operations do not change solutions to linear system.

Basic Algorithm:

1/ Put into echelon form working left to right.

2/ Put in reduced echelon form working right to left.

Example

$$\begin{pmatrix} 0 & 0 & 1 & 4 & -1 & 1 \\ 2 & -2 & 4 & 6 & 3 & 7 \\ 1 & -1 & 1 & -1 & 2 & 1 \end{pmatrix} \xrightarrow{\text{Switch } 1^{st} \text{ and } 3^{rd}} \begin{pmatrix} 1 & -1 & 1 & -1 & 2 & 1 \\ 2 & -2 & 4 & 6 & 3 & 7 \\ 0 & 0 & 1 & 4 & -1 & 1 \end{pmatrix}$$

Subtract 2×1^{st} from 2^{nd} $\downarrow$

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 2 & 1 \\ 0 & 0 & 2 & 8 & -1 & 5 \\ 0 & 0 & 1 & 4 & -1 & 1 \end{pmatrix} \xrightarrow{\text{Switch } 2^{nd} \text{ and } 3^{rd}} \begin{pmatrix} 1 & -1 & 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & 4 & -1 & 1 \\ 0 & 0 & 2 & 8 & -1 & 5 \end{pmatrix}$$

Subtract 2×2^{nd} from 3^{rd} $\downarrow$

Echelon Form

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & 4 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix} \xrightarrow[\text{then subtract } 2\times 3^{rd} \text{ from } 1]{\text{Add } 3^{rd} \text{ to } 2^{nd}} \begin{pmatrix} 1 & -1 & 1 & -1 & 0 & -5 \\ 0 & 0 & 1 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

Subtract 2^{nd} from 1^{st} $\downarrow$

Reduced Echelon Form $\rightarrow$

$$\begin{pmatrix} 1 & -1 & 0 & -5 & 0 & -9 \\ 0 & 0 & 1 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

Q: Once augmented matrix is in echelon form how do we find general solution?

<u>Definition</u> Let augmented matrix be in <u>echelon form</u>.

Pivot Column = Column with non-zero leading entry

Pivot Position = Location of leading non-zero entry.

Free Column = Non-pivot column of coefficient matrix.

<u>Example</u>

$$\begin{array}{cccccc} & & & & & \\ \left(\begin{array}{ccccc|c} 1 & -1 & 0 & -5 & 0 & -9 \\ 0 & 0 & 1 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right) \\ \uparrow\ \text{Pivot} \quad \uparrow\ \text{Free} \quad \uparrow\ \text{Pivot} \quad \uparrow\ \text{Free} \quad \uparrow\ \text{Pivot} \end{array}$$

O = Pivot Positions

Not free because not part of coefficient matrix.

Three Possibilities :

if and only if

1/ Last column is pivot $\Longleftrightarrow$ Inconsistent Linear System (No solutions)

<u>Example</u>

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 \end{array}\right) \Rightarrow \begin{aligned} x_1 + 3x_3 &= 4 \\ x_3 &= 3 \\ 0 &= 2 \end{aligned}$$

impossible

2/ Last column is not pivot <u>and</u> No free columns $\Longleftrightarrow$ Unique Solution

Found by backward substitution

<u>Example</u>

$$\left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 6 & 12 \end{array}\right) \Rightarrow \begin{aligned} x_1 - x_2 &= -1 \\ 6x_2 &= 12 \end{aligned} \Rightarrow \begin{aligned} x_1 &= 1 \\ x_2 &= 2 \end{aligned}$$

3/ Last column is not pivot **and** There are free columns $\iff$ Infinitely many solutions

Found by backward substitution of free variables

Example

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & -5 & 0 & -9 \\ 0 & 0 & 1 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right) \Rightarrow \begin{array}{r} x_1 - x_2 - 5x_4 = -9 \\ x_3 + 4x_4 = 4 \\ x_5 = 3 \end{array}$$

$$\Rightarrow \text{General Solution} = \begin{cases} x_2 + 5x_4 - 9 \\ x_2 \text{ free} \\ 4 - 4x_4 \\ x_4 \text{ free} \\ x_5 = 3 \end{cases}$$

Overview

Linear System

$\downarrow$ Represent with matrix

Augmented Matrix

$\downarrow$ Row Reduction

Echelon / Reduced Echelon Augmented Matrix

Last column is pivot $\rightarrow$ Inconsistent System

Last column not pivot:

- No free columns $\rightarrow$ Unique Solution
- Free Columns $\rightarrow$ Infinitely many solutions

$\underbrace{\text{Unique Solution} \qquad \text{Infinitely many solutions}}_{\text{Found by backwards substitution.}}$