Finite Symmetric Groups

Group under composition of functions

$\mathrm{Sym}_n = \Sigma(\{1,2,\dots,n\}) = \{\sigma : \{1,2,\dots,n\} \to \{1,2,\dots,n\} \mid \sigma \text{ a bijection}\}$

$|\mathrm{Sym}_n| = n!$

Aim: Find an effective way to describe $\sigma \in \mathrm{Sym}_n$.

Example: $n = 6$

cycles

cycle notation

$$\sigma : \begin{array}{l} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 1 \\ 4 \mapsto 5 \\ 5 \mapsto 4 \\ 6 \mapsto 6 \end{array} \Rightarrow 1 \to 2 \to 3, \quad 4 \to 5, \quad 6 \quad \Rightarrow \sigma = (123)(45)(6)$$

Exercise Any $\sigma \in \mathrm{Sym}_n$ can be written as product of disjoint cycles $(ab\dots f)$. Moreover any such decomposition is unique up to reordering the cycles and starting elements.

$a \to b \to \dots \to f$

Example $\sigma = (123)(45)(6) = (231)(6)(54)$

Cycle notations makes composition easier to understand. For example

Read this way

$$\begin{aligned} \sigma &= (123)(45)(6) \\ \tau &= (4351)(26) \end{aligned} \Rightarrow \quad \begin{aligned} \sigma \circ \tau &= (123)(45)(6)(4351)(26) \\ &= (152634) \end{aligned}$$

Observation: Given $\sigma \in \mathrm{Sym}_n$ and decomposition into disjoint cycles, the sum of the cycle lengths equals n.

Definition Let $\sigma \in \mathrm{Sym}_n$. The cycle structure of σ is the partition of n, given by the disjoint cycle lengths coming from σ.

Any way to breakdown n as a sum of natural numbers

Example: Cycle structure of $(123)(45)(6)$ is $\{1,2,3\}$

Convention: We generally omit cycles of length 1 from the notation

e.g. $(123)(45)(6) = (123)(45)$

The following important facts are proven in the notes.

$\exists \alpha \in Sym_n$ such that $\tau = \alpha \sigma \alpha^{-1}$

Proposition $\sigma, \tau \in Sym_n$ are conjugate $\iff$ σ, τ have same cycle type

So conjugacy classes are indexed by partitions of n

Proposition • $(a_1 a_2 \dots a_m) \in Sym_n \Rightarrow ord(a_1 \dots a_m) = m$

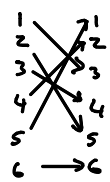
$k_i \in \mathbb{N}, \sum_{i=1}^{r} k_i = n$

• If $\sigma \in Sym_n$ has cycle structure $\{k_1, \dots, k_r\}$ then

$ord(\sigma) = LCM(k_1, \dots, k_r)$

Lowest common multiple

Examples $\sigma = (123)(45)(6)$ conjugate to $\tau = (354)(21)(6)$

If α :  , then $\alpha^{-1}\tau\alpha = \sigma \Rightarrow \tau = \alpha\sigma\alpha^{-1}$

$ord((123)(45)(6)) = LCM\{1,2,3\} = 6$

Just flips two elements

Definition A transposition is a cycle of length 2.

Theorem Any $\sigma \in Sym_n$ is the composition of transpositions. Moreover

$\sigma = \tau_1 \dots \tau_r = \alpha_1 \dots \alpha_s$, τ_i, α_j transpositions $\Rightarrow r \equiv s \mod 2$

Proof

$k-1$ transpositions

$(a_1 \dots a_k) = (a_1 a_k)(a_1 a_{k-1}) \dots (a_1 a_3)(a_1 a_2)$

$\Rightarrow$ Every $\sigma \in Sym_n$ can be written as a product of transpositions

We say $\sigma \in Sym_n$ is <u>even</u> if it has an <u>even number of even length cycles</u> in its cycle decomposition.

We say $\sigma \in Sym_n$ is <u>odd</u> if it has an <u>odd number of even length cycles</u> in its cycle decomposition.

For example, $(123)(45)(6)$ is odd as it has 1 even length cycle. Conversely $(123)(456)$ is even as it has 0 even length cycles.

<u>Claim</u> If τ is a transposition $(a_1 a_i)$ then

$$\sigma \text{ even} \Rightarrow \tau\sigma \text{ odd}$$

$$\sigma \text{ odd} \Rightarrow \tau\sigma \text{ even}$$

If a_1, a_i are in same cycle in σ

- $(a_1 a_i)(a_1 a_2 \dots a_i \dots a_r) = (a_1 a_2 \dots a_{i-1})(a_i a_{i+1} \dots a_r)$

If a_1, a_i in different cycles in σ

- $(a_1 a_i)(a_1 a_2 \dots a_{i-1})(a_i a_{i+1} \dots a_r) = (a_1 a_2 \dots a_r)$

In all possible cases, composing by τ either adds or removes exactly one even cycle.

For example, $\overbrace{(13)}^{\tau}\overbrace{(123)(45)(6)}^{\sigma} = (12)(3)(45)(6)$

$$(13)(12)(34)(5)(6) = (1234)(5)(6)$$

$e \in Sym_n$ is even $\Rightarrow \begin{cases} \text{Composition of even number of transpositions is even} \\ \text{Composition of odd number of transpositions is odd} \end{cases}$

$\Rightarrow \sigma = \tau_1 \dots \tau_r = \alpha_1 \dots \alpha_s$, τ_i, α_j transpositions $\Rightarrow r \equiv s \bmod 2$

$\square$

Observations

- $e \in \mathrm{Sym}_n$ even
- σ, τ even $\Rightarrow$ $\sigma\tau$ even
- σ even $\Rightarrow$ σ^{-1} even

Definition The Alternating subgroup of Sym_n is the subgroup

$$\mathrm{Alt}_n := \{\sigma \in \mathrm{Sym}_n \mid \sigma \text{ even}\}$$

Proposition $(\mathrm{Sym}_n : \mathrm{Alt}_n) = 2$ and Alt_n is generated by all cycles of length 3.

Proof Let $\tau \in \mathrm{Sym}_n$ be a transposition.

$\tau \mathrm{Alt}_n = \{\sigma \in \mathrm{Sym}_n \mid \sigma \text{ odd}\}$

$\Rightarrow$ $\mathrm{Sym}_n = \mathrm{Alt}_n \amalg \tau \mathrm{Alt}_n$ $\Rightarrow$ $(\mathrm{Sym}_n : \mathrm{Alt}_n) = 2$

$\sigma \in \mathrm{Alt}_n \Rightarrow$ $\sigma = \tau_1 \tau_2 \cdots \tau_r$, τ_i transpositions, r even.

Observe

$$(ij)(kl) = (k\,i\,l)(i\,j\,k) \quad \text{and} \quad (ij)(ik) = (i\,k\,j)$$

$\Rightarrow$ σ can be expressed as a composition of cycles of length 3.

All cycles of length 3 are even hence Alt_n is generated by them

$\square$