Subspaces, Kernels and Ranges

Definition A **subspace** of a vector space V is a subset $U \subset V$ such that

contained in (could be equal)

1/ $\underline{0}$ in U

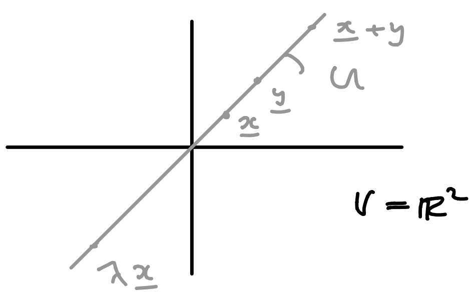
2/ $\underline{x}, \underline{y}$ in $U \Rightarrow \underline{x}+\underline{y}$ in U

3/ $\underline{x}$ in $U \Rightarrow \lambda \underline{x}$ in U

Remark A subspace is a vector space in its own right.

Examples

1/



U is a subspace of $\mathbb{R}^2$

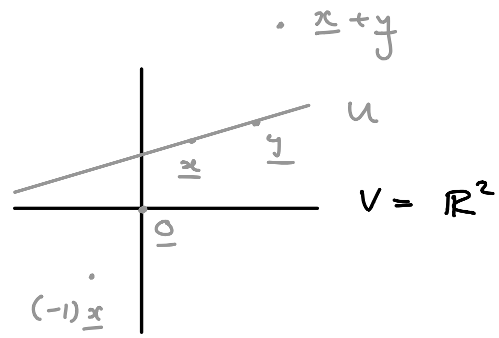
2/ All solutions to $y'' + y = 0$ in $\mathbb{C}^1(\mathbb{R})$.

1/ $0'' + 0 = 0$ ← zero function

2/ $f'' + f = 0$ and $g'' + g = 0 \Rightarrow (f+g)'' + (f+g) = 0$

3/ $f'' + f = 0 \Rightarrow (\lambda f)'' + \lambda f = \lambda(f'' + f) = 0$

3/ (Non-example)



1/, 2/, and 3/ fail

$\Downarrow$

U <u>not</u> a subspace of V

Let $T : V \to W$ be a linear transformation between V and W, two vector spaces.

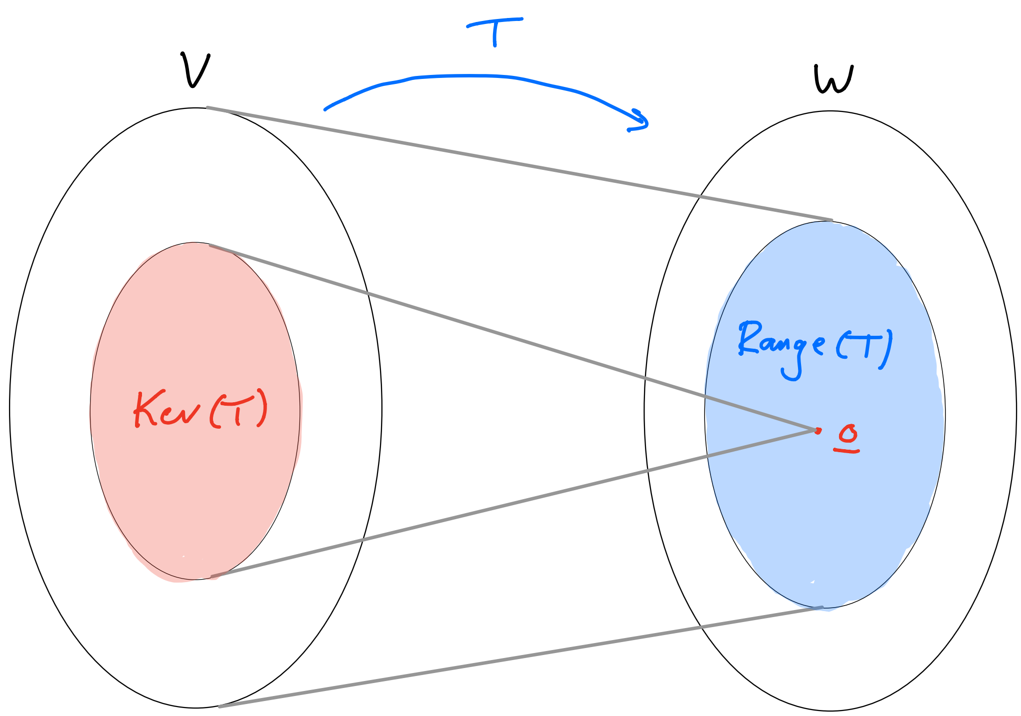
<u>Definition</u>

kernel of T

zero vector in W

$$\mathrm{Ker}(T) := \{ \underline{x} \text{ in } V \text{ such that } T(\underline{x}) = \underline{0} \}$$

$$\mathrm{Range}(T) := \{ T(\underline{x}) \text{ in } W \text{ such that } \underline{x} \text{ in } V \}$$



<u>Theorem</u> $\mathrm{Ker}(T) \subset V$ <u>and</u> $\mathrm{Range}(T) \subset W$ are subspaces.

<u>Proof</u> $\mathrm{Ker}(T) \subset V$ a subspace

1/ $T(\underset{\text{in } V}{\underline{0}}) = T(0 \cdot \underline{0}) = 0 \cdot T(\underline{0}) = \underset{\text{in } W}{\underline{0}}$

$\Rightarrow \underline{0}$ in $\ker(T)$

2/ $\underline{u}, \underline{v}$ in $\ker(T)$ $\Rightarrow T(\underline{u}+\underline{v}) = T(\underline{u}) + T(\underline{v})$
$= \underline{0} + \underline{0} = 0$
$\Rightarrow \underline{u}+\underline{v}$ in $\ker(T)$

3/ $\underline{u}$ in $\ker(T)$, λ in $\mathbb{R}$ $\Rightarrow T(\lambda \underline{u}) = \lambda T(\underline{u}) = \lambda \cdot \underline{0} = \underline{0}$
$\Rightarrow \lambda \underline{u}$ in $\ker(T)$

$\mathrm{Range}(T) \subset W$ a subspace

1/ $T(\underline{0}) = \underline{0} \Rightarrow \underline{0}$ in $\mathrm{Range}(T)$

2/ $\underline{u}, \underline{v}$ in $\mathrm{Range}(T)$ $\Rightarrow \underline{u} = T(\underline{x}),\ \underline{v} = T(\underline{y})$
for some $\underline{x}, \underline{y}$ in V

$\Rightarrow \underline{u}+\underline{v} = T(\underline{x}) + T(\underline{y}) = T(\underline{x}+\underline{y})$

$\Rightarrow \underline{u}+\underline{v}$ in $\mathrm{Range}(T)$

3/ $\underline{u}$ in $\mathrm{Range}(T)$, λ in $\mathbb{R}$ $\Rightarrow \underline{u} = T(\underline{x})$ for some $\underline{x}$ in V
$\Rightarrow \lambda \underline{u} = \lambda T(\underline{x}) = T(\lambda \underline{x})$
$\Rightarrow \lambda \underline{u}$ in $\mathrm{Range}(T)$ $\square$

Means proof is done

If $T_A : \mathbb{R}^n \to \mathbb{R}^m$ where $A = (\underline{a}_1 \dots \underline{a}_n)$ in $\mathbb{R}^m$

$\underline{x} \longrightarrow A\underline{x}$

$\ker(T_A) = \{\underline{x}$ in $\mathbb{R}^n$ such that $A\underline{x} = \underline{0}\} = \text{Null}(A)$

All solutions to homogeneous linear system $A\underline{x} = 0$

The null space of A

$\text{Range}(T_A) = \{A\underline{x}$ such that $\underline{x}$ in $\mathbb{R}^n\} = \text{Span}(\underline{a}_1, \dots, \underline{a}_n)$

$\quad = \text{Col}(A)$

Column Space of A

__Calculus Example__

functions with continuous derivative

$T : \mathbb{C}^1(\mathbb{R}) \longrightarrow C(\mathbb{R})$ Continuous Functions

$y \longmapsto y'' + y$

__Claim__ T linear

1) $T(f+g) = (f+g)'' + (f+g) = (f''+f) + (g''+g)$

$= T(f) + T(g)$

2) $T(\lambda f) = (\lambda f)'' + \lambda f = \lambda(f'' + f) = \lambda T(f)$

$\square$

$\ker(T) = \{f$ in $\mathbb{C}^1(\mathbb{R})$ such that $f'' + f = 0\}$

$\Rightarrow \ker(T)$ is the set of all solutions to the differential equation $y'' + y = 0$