Subrings, Ideals and Homomorphisms

<u>Definition</u> Let R be a ring. A subset $S \subset R$ is a <u>subring</u> if

1/ $0_R \in S$
2/ $x, y \in S \Rightarrow x + y \in S$
3/ $x \in S \Rightarrow -x \in S$

S a subgroup of R under addition

4/ $1_R \in S$
5/ $x, y \in S \Rightarrow xy \in S$

<u>Examples</u>: $\mathbb{Z} \subset \mathbb{Q}$, $\mathbb{Q} \subset \mathbb{R}$, $\mathbb{R} \subset \mathbb{C}$, $M_n(\mathbb{Z}) \subset M_n(\mathbb{R})$

<u>Non-example</u> : $m\mathbb{Z} \subset \mathbb{Z}$ does not contain 1

<u>Remark</u> A subring is a ring under the induced + and ×

<u>Definition</u> Let R and T be rings. A <u>homomorphism</u> from R to T is a map $\phi : R \longrightarrow T$ such that

1/ $\phi(\underset{\text{in } R}{x+y}) = \phi(x) \underset{\text{in } T}{+} \phi(y) \quad \forall x, y \in R$
2/ $\phi(\underset{\text{in } R}{xy}) = \phi(x) \underset{\text{in } T}{\cdot} \phi(y) \quad \forall x, y \in R$
3/ $\phi(1_R) = 1_T$

<u>Remark</u>

1/ $\Rightarrow \phi$ is a group homomorphism from $(R, +)$ to $(T, +)$

Isomorphism (of rings) = Bijective homomorphism (of rings)

$R \cong T \iff \exists\, \phi : R \to T$ an isomorphism

Proposition

$\phi : R \longrightarrow T$ a homomorphism $\Rightarrow$ $\mathrm{Im}\,\phi \subset T$ a subring

Proof ϕ a group homomorphism from $(R,+)$ to $(T,+)$

$\Rightarrow$ $\mathrm{Im}(\phi) \subset T$ a subgroup under addition.

Let $\phi(x), \phi(y) \in \mathrm{Im}\,\phi \Rightarrow \phi(x)\phi(y) = \phi(xy)$

$\Rightarrow \phi(x)\phi(y) \in \mathrm{Im}\,\phi$ Closure under $\times$

$\phi(1_R) = 1_T \Rightarrow 1_T \in \mathrm{Im}\,\phi$

$\square$

Definition

$\phi : R \longrightarrow T$ a homomorphism (of rings)

$\mathrm{Ker}\,\phi = \{x \in R \mid \phi(x) = 0_T\}$ additive identity

Remark: $\ker\phi = \{0_R\} \Leftrightarrow \phi$ injective ($\Rightarrow R$ isomorphic to $\mathrm{Im}\,\phi$)

Q: Is $\mathrm{Ker}\,\phi \subset R$ a subring?

Observation: $1_R \in \mathrm{Ker}\,\phi \Rightarrow \phi(1_R) = 0_T \Rightarrow 0_T = 1_T$

$\Rightarrow T$ trivial ← Very restrictive

If it's not a subring in general, what is it.

Observations

$\phi : R \to T$ homomorphism of groups under addition

$\Rightarrow \mathrm{Ker}\,\phi \subset R$ is a subgroup under $+$

Let $x \in \mathrm{Ker}\,\phi$, $y \in R$ ← not necessarily in $\mathrm{Ker}\,\phi$

$\phi(xy) = \phi(x)\phi(y) = 0_T\,\phi(y) = 0_T \Rightarrow xy \in \mathrm{Ker}\,\phi$

$\phi(yx) = \phi(y)\phi(x) = \phi(y)\,0_T = 0_T \Rightarrow yx \in \mathrm{Ker}\,\phi$

Conclusion $\ker\phi \subset R$ is closed under left and right multiplication by <u>all of R</u>.

Definition Let R be a ring. An <u>ideal</u> of R is a subset $I \subset R$ such that

1/ I is a subgroup under $+$

2/ $x \in I,\ r \in R \Rightarrow rx, xr \in I$

Remark $\ker\phi \subset R$ is an ideal.

Let $I \subset R$ be an ideal.

$R/I = \{x + I \mid x \in R\}$ = left cosets under $+$ of I in R.

Theorem The binary operations

$$R/I \times R/I \longrightarrow R/I \qquad\qquad R/I \times R/I \longrightarrow R/I$$
$$(x+I, y+I) \rightarrow (x+y)+I \qquad\qquad (x+I, y+I) \rightarrow (xy)+I$$

addition multiplication

are well-defined and give R/I the structure of a ring

quotient ring

Proof (Outline)

$(R, +)$ Abelian $\Rightarrow$ $(I, +)$ is a normal subgroup of $(R, +)$

$\Rightarrow$ addition on R/I well defined

Claim :

$$\left.\begin{array}{l} x_1 + I = x_2 + I \\ y_1 + I = y_2 + I \end{array}\right\} \Rightarrow \quad x_1 y_1 + I = x_2 y_2 + I$$

Subproof

$$\left.\begin{array}{l} x_1 + I = x_2 + I \\ y_1 + I = y_2 + I \end{array}\right\} \Rightarrow \quad x_1 - x_2 ,\ y_1 - y_2 \in I$$

I an ideal of R

$\Rightarrow \quad (x_1 - x_2) y_1 - x_2 (y_1 - y_2) \in I$

$\Rightarrow \quad x_1 y_1 - x_2 y_2 \in I \Rightarrow x_1 y_1 + I = x_2 y_2 + I$

$\Rightarrow \quad \times$ on R/I well-defined

The rest is an exercise. For example,

$$0_{R/I} = 0_R + I \quad \text{and} \quad 1_{R/I} = 1_R + I$$

$\square$

1st Isomorphism Theorem (for Rings)

Let $\phi : R \longrightarrow T$ be a ring homomorphism then

$$\begin{aligned} \psi : R/\mathrm{Ker}\,\phi &\longrightarrow \mathrm{Im}\,\phi \\ x + \mathrm{Ker}\,\phi &\longrightarrow \phi(x) \end{aligned}$$

is a ring isomorphism.

Proof From group theory we know ψ is an isomorphism of additive groups.

$$\psi((x + \mathrm{Ker}\,\phi)(y + \mathrm{Ker}\,\phi)) = \psi((xy) + \mathrm{Ker}\,\phi)$$

$= \phi(xy) = \phi(x)\phi(y) = \psi(x + \operatorname{Ker}\phi)\psi(y + \operatorname{Ker}\phi)$

$\psi(1_R + \operatorname{Ker}\phi) = \phi(1_R) = 1_T$ $\square$