# Spanning, Linear Independence and Dimension

$V$ - vector space (e.g. $\mathbb{R}^n$, $\mathbb{C}(\mathbb{R})$, $\mathbb{P}_n(\mathbb{R})$)

$\{\underline{v}_1, \dots, \underline{v}_p\} \subset V$

Just a vector in $V$

in $\mathbb{R}$

$\lambda_1 \underline{v}_1 + \lambda_2 \underline{v}_2 + \dots + \lambda_p \underline{v}_p$ = Linear combination of $\{\underline{v}_1, \dots, \underline{v}_p\}$.

**Definition**

$\text{Span}(\underline{v}_1, \dots, \underline{v}_p)$ = Subset of all linear combinations of $\{\underline{v}_1, \dots, \underline{v}_p\}$.

E.g. $\text{Span}(\sin(x), \cos(x)) = \{\lambda_1 \sin(x) + \lambda_2 \cos(x)\} \subset \mathbb{C}(\mathbb{R})$

We say $\{\underline{v}_1, \dots, \underline{v}_p\}$ is a <u>spanning set</u> of $V$ if

$\text{Span}(\underline{v}_1, \dots, \underline{v}_p) = V$.

E.g. $\{\underline{e}_1, \dots, \underline{e}_n\} \subset \mathbb{R}^n$, $\{1, x, x^2, \dots, x^n\} \subset \mathbb{P}_n(\mathbb{R})$

<u>Fact</u>: $\text{Span}(\underline{v}_1, \dots, \underline{v}_p) \subset V$ is a <u>subspace</u>.

<u>Definition</u> We say $V$ is <u>finite dimensional</u> if there exists a <u>finite spanning set</u>. If <u>not</u>, we say $V$ is <u>infinite dimensional</u>.

E.g. $\mathbb{R}^n$, $\mathbb{P}_n(\mathbb{R})$ are finite dimensional

$\mathbb{C}(\mathbb{R})$, $\mathbb{P}(\mathbb{R})$ are infinite dimensional.

Q: If $V$ is finite dimensional what is its <u>dimension</u>?

**Definition**

(L.I.)

$\{\underline{v}_1, \dots, \underline{v}_p\} \subset V$ is linearly independent $\iff$

$\lambda_1 \underline{v}_1 + \dots + \lambda_p \underline{v}_p = \underline{0} \Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_p = 0$

(L.D.)

$\{\underline{v}_1, \dots, \underline{v}_p\} \subset V$ linearly dependent $\iff$

There exist $\lambda_1, \dots, \lambda_p$ not all zero such that $\lambda_1 \underline{v}_1 + \dots + \lambda_p \underline{v}_p = \underline{0}$

**Example** $\{\sin(x), \cos(x)\} \subset \mathbb{C}(\mathbb{R})$ L.I. ?

equality of functions (true for all $x$)

Assume $\lambda_1 \sin(x) + \lambda_2 \cos(x) = 0$

zero function

$\Rightarrow \lambda_1 \tan(x) + \lambda_2 = 0$

constant

If $\lambda_1 \neq 0$ then $\tan(x) = \dfrac{-\lambda_2}{\lambda_1}$ for all $x$. Contradiction.

$\Rightarrow \lambda_1 = 0 \Rightarrow \lambda_2 = 0 \Rightarrow \{\sin(x), \cos(x)\} \subset \mathbb{C}(\mathbb{R})$ L.I.

**Fact** : $\{\underline{v}_1, \dots, \underline{v}_p\}$ L.D.

$\Leftrightarrow$ $\underline{v}_j$ in Span $(\underline{v}_1, \dots, \underline{v}_{j-1}, \underline{v}_{j+1}, \dots, \underline{v}_p)$ for some $j$

$\Leftrightarrow$ Span $(\underline{v}_1, \dots, \underline{v}_{j-1}, \underline{v}_{j+1}, \dots, \underline{v}_p)$ = Span $(\underline{v}_1, \dots, \underline{v}_p)$

in span of others

E.g. $\text{Span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}\right) = \text{Span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right)$

**Fundamental Definition** Let $V$ be a finite dimensional vector space. A basis for $V$ is a subset $\{\underline{v}_1, \dots, \underline{v}_p\}$ such that

1. Span $(\underline{v}_1, \dots, \underline{v}_p) = V$
2. $\{\underline{v}_1, \dots, \underline{v}_p\}$ L.I.

**Examples**

Called the standard basis of $\mathbb{R}^n$

$\{\underline{e}_1, \dots, \underline{e}_n\}$ in $\mathbb{R}^n$, $\{1, x, x^2, \dots, x^n\}$ in $\mathbb{P}_n(\mathbb{R})$

$\{\sin(x), \cos(x)\}$ in $V = \{$ Solutions to $y'' + y = 0\}$

**Remark** Bases are <u>not</u> unique. For example

$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ <u>and</u> $\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right\}$ are both bases for $\mathbb{R}^2$.

**Important Facts** ($V$ a finite dimensional vector space)

A/ If $\{\underline{v}_1, \dots, \underline{v}_p\}$ is spanning set of $V$ then it contains a basis

E.g. $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ contains basis $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^2$

B/ If $\{\underline{v}_1, \dots, \underline{v}_p\}$ is L.I. then it can be extended to a basis.

in many ways

E.g. $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ L.I. in $\mathbb{R}^3$. $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ basis of $\mathbb{R}^3$

C/ If $\{\underline{v}_1, \dots, \underline{v}_p\}$ is L.I. and $\{\underline{u}_1, \dots, \underline{u}_k\}$ is spanning

$\Rightarrow \quad p \leq k$ (Linearly independent sets are smaller than or equal to spanning sets)

**Theorem** Any finite dimensional vector space has a basis.

**Proof** Choose $\{\underline{v}_1, \dots, \underline{v}_p\}$ a spanning set of $V$. By A/ it contains a basis

$\square$

**Theorem** Let $V$ be a finite dimensional vector space. Any two bases have same size.

**Proof** Let $\{\underline{v}_1, \dots, \underline{v}_p\}$ and $\{\underline{u}_1, \dots, \underline{u}_k\}$ be bases for $V$

$\{\underline{v}_1, \dots, \underline{v}_p\}$ is L.I. and $\{\underline{u}_1, \dots, \underline{u}_k\}$ is spanning $\Rightarrow p \leq k$

$\{\underline{u}_1, \dots, \underline{u}_k\}$ is L.I. and $\{\underline{v}_1, \dots, \underline{v}_p\}$ is spanning $\Rightarrow k \leq p$

$\Rightarrow p = k$

$\square$

**Definition** Let $V$ be a finite dimensional vector space. The **dimension** of $V$, denoted $\dim(V)$, is the size of a basis.

**Examples** $\dim(\mathbb{R}^n) = n$ ← Thank goodness! , $\dim(\mathbb{P}_n(\mathbb{R})) = n+1$

$V$ = plane in $\mathbb{R}^3$ containing $\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}$

$\Rightarrow \quad V = \mathrm{Span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right)$ , $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$ L.I

$\Rightarrow \quad \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$ basis for $V \Rightarrow \dim(V) = 2$

**Important Intuitions** ( $V$ finite dimensional )

$\{\underline{v}_1, \dots, \underline{v}_p\}$ L.I. : B/ $\Rightarrow \quad p \leq \dim(V)$

$\{\underline{v}_1, \dots, \underline{v}_p\}$ Spanning : A/ $\Rightarrow \quad p \geq \dim(V)$

$p = \dim(V)$ then $(\underline{v}_1, \dots, \underline{v}_p)$ L.I $\underset{A/, B/}{\Leftrightarrow}$ $\{\underline{v}_1, \dots, \underline{v}_p\}$ Spanning

**Conclusion** :

Basis = Minimal Spanning Set = Maximal L.I. Set

**Familiar Example**

$\{\underline{a}_1, \dots, \underline{a}_n\}$ in $\mathbb{R}^n$ , L.I $\Leftrightarrow$ Reduced $A = (\underline{a}_1 \dots \underline{a}_n)$ has pivot in every column

$\Leftrightarrow$ Reduced $A = (\underline{a}_1 \dots \underline{a}_n)$ has pivot in every row

$\Leftrightarrow$ $\mathrm{Span}(\underline{a}_1, \dots, \underline{a}_n) = \mathbb{R}^n$

**Theorem** Let $V$ be a finite dimensional vector space, with $U \subset V$ a subspace. Then $U$ is also finite dimensional and $\dim(U) \leq \dim(V)$ with equality $\Leftrightarrow U = V$.