Rank and Nullity

V, W vector spaces

$T : V \to W$ linear transformation

$\mathrm{Ker}(T) = \{ \underline{x} \text{ in } V \text{ such that } T(\underline{x}) = \underline{0} \} \subset V$ subspace

$\mathrm{Range}(T) = \{ T(\underline{x}) \text{ in } W \text{ such that } \underline{x} \text{ in } V \} \subset W$ subspace

Important Example:

A – $m \times n$ matrix

$\mathrm{Ker}(T_A) = \mathrm{Null}(A) = \{ \underline{x} \text{ in } \mathbb{R}^n \text{ such that } A\underline{x} = \underline{0} \}$

$\mathrm{Range}(T_A) = \mathrm{Col}(A) = \{ A\underline{x} = x_1 \underline{a}_1 + \dots + x_n \underline{a}_n \} = \mathrm{Span}(\underline{a}_1, \dots, \underline{a}_n)$

$\mathrm{Ker}(T_A) = \{\underline{0}\} \iff \{\underline{a}_1, \dots, \underline{a}_n\}$ L.I.

The bigger $\mathrm{Ker}(T_A)$ the more linearly dependent the columns.

V finite dimensional

Fact : $\dim(V) < \infty \Rightarrow \dim(\mathrm{Ker}(T)) < \infty, \ \dim(\mathrm{Range}(T)) < \infty$

Definition

$\mathrm{Rank}(T) = \dim(\mathrm{Range}(T))$

$\mathrm{Nullity}(T) = \dim(\mathrm{Ker}(T))$

Example

$$A = \begin{array}{c} \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 & 1 \\ -1 & -2 & 0 & -1 & 0 \end{pmatrix} \\ \begin{array}{ccccc} \| & \| & \| & \| & \| \\ \underline{a}_1 & \underline{a}_2 & \underline{a}_3 & \underline{a}_4 & \underline{a}_5 \end{array} \end{array} \longrightarrow \begin{pmatrix} \boxed{1} & 2 & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \underline{a}_2 \text{ in } \mathrm{Span}(\underline{a}_1), \ \underline{a}_4 \text{ in } \mathrm{Span}(\underline{a}_1, \underline{a}_2, \underline{a}_3)$

$\Rightarrow \mathrm{Range}(T_A) = \mathrm{Span}(\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4, \underline{a}_5) = \mathrm{Span}(\underline{a}_1, \underline{a}_3, \underline{a}_5)$

Pivot position in $1^{st}, 3^{rd}, 5^{th}$ column $\Rightarrow \{\underline{a}_1, \underline{a}_3, \underline{a}_5\}$ L.I.

$\Rightarrow \{\underline{a}_1, \underline{a}_3, \underline{a}_5\}$ basis $\Rightarrow \mathrm{Rank}(T_A) = 3$

Fact : Pivot Columns of a matrix A form a basis for $\mathrm{Range}(T_A) / \mathrm{Col}(A)$

$\Rightarrow \mathrm{Rank}(T_A) =$ Number of Pivot columns of A

$\mathrm{Ker}(T_A) = \{ \underline{x} \text{ in } \mathbb{R} \text{ such that } A\underline{x} = \underline{0} \}$

x_2, x_4 free

$\Rightarrow$ General solution to $A\underline{x} = 0$ is
$$\begin{pmatrix} -2x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \mathrm{Ker}(T_A) = \mathrm{Span}\left(\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right).$$

Observe

$$x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad x_2 = x_4 = 0 \quad \Rightarrow \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\} \text{ L.I.}$$

$$\Rightarrow \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\} \text{ a basis for } \mathrm{Ker}(T_A).$$

$\Rightarrow \quad \mathrm{Nullity}(T_A) = 2$

<u>Fact</u> : This procedure always give a basis for $\mathrm{Ker}(T_A)$

$\Rightarrow$ $\mathrm{Nullity}(T_A)$ = Number of free columns of A.

Observation : (A an $m \times n$ matrix)

Number of columns of A = Number of Pivot columns of A

+ Number of free columns of A.

$\Rightarrow \quad n = \mathrm{Rank}(T_A) + \mathrm{Nullity}(T_A)$

This is a special case of the following result

<u>Theorem</u> $T: V \to W$ linear, $\dim(V) < \infty$. Then

$$\dim V = \underset{\underset{\dim(\mathrm{Range}(T))}{\parallel}}{\mathrm{Rank}(T)} + \underset{\underset{\dim(\mathrm{Ker}(T))}{\parallel}}{\mathrm{Nullity}(T)}$$