Math 1B: Calculus **Spring 2020**

Project 1.3: The Abel and Dirichlet Tests

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Whilst we have a large selection of series tests, there are still examples of infinite series that fall outside of their reach. In this project we'll look at two powerful new tests.

Abel's Test. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences. Suppose the following statements are true:*

- $\sum_{n=1}^{\infty} a_n$ *is convergent.*
- $\{b_n\}$ *is bounded and monotonic.*

Then $\sum_{n=1}^{\infty} a_n b_n$ *is convergent.*

Use this test to prove the convergence of the following series:

1. $$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \cos(\frac{1}{n})$$

   Carefully justify your answer.

2.
$$\sum_{n=2}^{\infty}\left(\frac{2^n+3^n}{4^n+5^n}\cdot\sum_{k=1}^{n}\frac{1}{k^2}\right)$$

Carefully justify your answer.

Dirichlet's Test. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences. Suppose the following statements are true:*

- *The sequence* $\{a_n\}$ *is decreasing and* $\lim_{n\to\infty} a_n = 0$
- *The sequence of partial sums of* $\{b_n\}$ *is bounded.*

Then $\sum_{n=1}^{\infty} a_n b_n$ *is convergent.*

3. Show that Dirichlet's Test implies the Alternating Series Test.

   Carefully justify your answer.

4. Show the following series converges

$$\sum_{n=1}^{\infty} \frac{\cos(n)}{n}.$$

Hint: To simplify the partial sum $\sum_{n=1}^{N} \cos(n)$, try multiplying by $2\sin(\frac{1}{2})$ and using the formula $2\sin(b)\cos(a) = \sin(a+b) - \sin(a-b)$.

Bonus Problem: Is it absolutely or conditionally convergent?