# Orthogonal Sets

Recall : $\underline{u}, \underline{v}$ in $\mathbb{R}^n$ are orthogonal $\iff$ $\underline{u} \cdot \underline{v} = 0$

## Definition

$\{\underline{u}_1, \dots, \underline{u}_p\} \subset \mathbb{R}^n$ is an orthogonal set $\iff$ $\underline{u}_i \cdot \underline{u}_j = 0 \ \forall \ i \neq j$.

$\{\underline{u}_1, \dots, \underline{u}_p\} \subset \mathbb{R}^n$ is an orthonormal set $\iff$ $\underline{u}_i \cdot \underline{u}_j = 0 \ \forall \ i \neq j$ and $\|\underline{u}_i\| = 1$ for all $i$

(ie an orthogonal set of unit vectors)

**Theorem** Let $\{\underline{u}_1, \dots, \underline{u}_p\}$ be an orthogonal set of non-zero vectors.

$$\underline{v} = \lambda_1 \underline{u}_1 + \dots + \lambda_p \underline{u}_p \implies \lambda_i = \frac{\underline{v} \cdot \underline{u}_i}{\underline{u}_i \cdot \underline{u}_i}$$

**Proof** $\underline{v} = \lambda_1 \underline{u}_1 + \dots + \lambda_p \underline{u}_p \implies \underline{v} \cdot \underline{u}_i = (\lambda_1 \underline{u}_1 + \dots + \lambda_p \underline{u}_p) \cdot \underline{u}_i$

$$\implies \underline{v} \cdot \underline{u}_i = \lambda_1 \underline{u}_1 \cdot \underline{u}_i + \dots + \lambda_i \underline{u}_i \cdot \underline{u}_i + \dots + \lambda_p \underline{u}_p \cdot \underline{u}_i$$

$\underline{u}_i \neq \underline{0} \implies \underline{u}_i \cdot \underline{u}_i \neq 0$

$$= \lambda_i \underline{u}_i \cdot \underline{u}_i$$

$$\implies \lambda_i = \frac{\underline{v} \cdot \underline{u}_i}{\underline{u}_i \cdot \underline{u}_i} \qquad \square$$

**Consequence** : $\{\underline{u}_1, \dots, \underline{u}_p\}$ orthogonal set of non-zero vectors $\implies$ L.I.

$$\left( \underline{0} = \lambda_1 \underline{u}_1 + \dots + \lambda_p \underline{u}_p \implies \lambda_i = \frac{\underline{0} \cdot \underline{u}_i}{\underline{u}_i \cdot \underline{u}_i} = 0 \right)$$

**Examples** Determine $\begin{pmatrix} 1 \\ 2 \end{pmatrix}_{\mathcal{B}}$ where $\mathcal{B} = \left\{ \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$

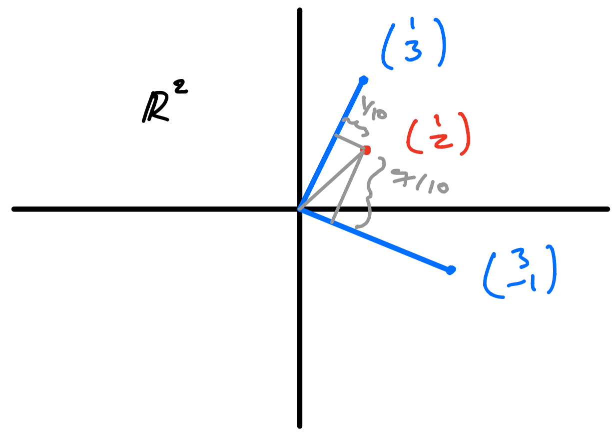
Observe $\left\{ \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$ is an orthogonal set of $\mathbb{R}^2$.

If $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \lambda_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$$\implies \lambda_1 = \frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix}}{\begin{pmatrix} 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix}} = \frac{1 \cdot 3 + 2 \cdot (-1)}{3^2 + (-1)^2} = \frac{1}{10}, \quad \lambda_2 = \frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}} = \frac{7}{10}$$

$\Rightarrow \quad \begin{pmatrix}1\\2\end{pmatrix}_\beta = \begin{pmatrix}1/10\\7/10\end{pmatrix}$

Visualization :




Definition

$\{\underline{u}_1, \dots, \underline{u}_n\} \subset \mathbb{R}^n$ is an orthogonal basis $\Leftrightarrow$ 1/ It is an orthogonal set 2/ It is a basis.

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Main Example : $\{\underline{e}_1, \dots, \underline{e}_n\} \subset \mathbb{R}^n$ is an orthonormal basis

$\left\{ \begin{pmatrix}3\\-1\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix} \right\}$ = orthogonal basis of $\mathbb{R}^2$

$\leftarrow$ normalization

$\left\{ \begin{pmatrix}3/\sqrt{10}\\-1/\sqrt{10}\end{pmatrix}, \begin{pmatrix}1/\sqrt{10}\\3/\sqrt{10}\end{pmatrix} \right\}$ = orthonormal basis of $\mathbb{R}^2$

Important Observation :

$A = (\underline{a}_1 \dots \underline{a}_n)$ — $m \times n$ matrix (in $\mathbb{R}^m$) $\Rightarrow A^T = \begin{pmatrix}\underline{a}_1^T\\ \vdots \\ \underline{a}_n^T\end{pmatrix}$ — $n \times m$ matrix

$$\Rightarrow A^T A = \begin{pmatrix} \underline{a}_1^T\underline{a}_1 & \underline{a}_1^T\underline{a}_2 & \dots & \underline{a}_1^T\underline{a}_n \\ \underline{a}_2^T\underline{a}_1 & \underline{a}_2^T\underline{a}_2 & \dots & \underline{a}_2^T\underline{a}_n \\ \vdots & & & \vdots \\ \underline{a}_n^T\underline{a}_1 & \dots & \dots & \underline{a}_n^T\underline{a}_n \end{pmatrix} \Rightarrow (A^TA)_{ij} = \underline{a}_i \cdot \underline{a}_j$$

Consequences : Let $U = (\underline{u}_1 \dots \underline{u}_n)$ — $m \times n$ matrix (in $\mathbb{R}^m$)

1/ $\{\underline{u}_1, \dots, \underline{u}_n\}$ orthogonal set $\Leftrightarrow$ $U^TU$ diagonal

2/ $\{\underline{u}_1, \dots, \underline{u}_n\}$ orthonormal set $\Leftrightarrow$ $U^T U = I_n$

3/ $\{\underline{u}_1, \dots, \underline{u}_n\}$ orthonormal set

$$\Rightarrow \quad (U\underline{x}) \cdot (U\underline{y}) = (U\underline{x})^T U\underline{y} = \underline{x}^T U^T U \underline{y} = \underline{x}^T \underline{y} = \underline{x} \cdot \underline{y}$$

length in $\mathbb{R}^n$

4/ $(U\underline{x}) \cdot (U\underline{x}) = \underline{x} \cdot \underline{x} \Rightarrow \|U\underline{x}\| = \|\underline{x}\|$ Length in $\mathbb{R}^n$

3/, 4/ $\Rightarrow$ $T_U : \mathbb{R}^n \to \mathbb{R}^n$, $\underline{x} \to U\underline{x}$ preserves the standard inner product, hence lengths and distances.

5/ $U = (\underline{u}_1 \dots \underline{u}_n)$ — $n \times n$ matrix (in $\mathbb{R}^n$)

$\{\underline{u}_1, \dots, \underline{u}_n\}$ an <u>orthonormal basis</u> $\Leftrightarrow$ $U^T U = I_n \Leftrightarrow U^T = U^{-1}$

We call such a matrix an <u>orthogonal matrix</u>.

<u>Key Fact</u>: $U$ — $n \times n$ an orthogonal matrix $\Leftrightarrow$ $T_U : \mathbb{R}^n \to \mathbb{R}^n$ preserves the standard inner product.
ie $(U\underline{x}) \cdot (U\underline{y}) = \underline{x} \cdot \underline{y}$ for all $\underline{x}, \underline{y}$ in $\mathbb{R}^n$.