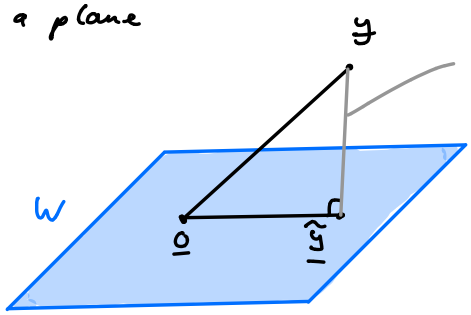
# Orthogonal Projections

$W \subset \mathbb{R}^n$ a subspace, $\underline{y}$ a vector in $\mathbb{R}^n$.

Q) What is the <u>minimum distance</u> between $\underline{y}$ and $W$ ?

Example : $W \subset \mathbb{R}^3$ a plane




$\|y - \hat{y}\|$ = min distance between $\underline{y}$ and $W$

Key Observation : $\underline{y} - \underline{\hat{y}}$ orthogonal to every vector in $W$

$$\Rightarrow \quad y - \hat{y} = W^{\perp}$$

<u>Aim</u> : Find $\hat{y}$ in $W$ such that $y - \hat{y}$ in $W^{\perp}$.

Let $\{\underline{u}_1, \ldots, \underline{u}_p\}$ be an <u>orthogonal basis</u> for $W$.

$\Rightarrow \quad \mathrm{Span}(\underline{u}_1, \ldots, \underline{u}_p) = W,$

$\underline{u}_i \cdot \underline{u}_j = 0 \quad \forall \ i \neq j$ and $\underline{u}_i \cdot \underline{u}_i \neq 0$ for all $i$ — The $\underline{u}_i$ are non-zero

Need $\lambda_1, \ldots, \lambda_p$ such that

1) $\hat{y} = \lambda_1 \underline{u}_1 + \cdots + \lambda_p \underline{u}_p$ — $\hat{y}$ in $W$

2) $(\underline{y} - \hat{y}) \cdot \underline{u}_i = 0$ for all $i$. — $y - \hat{y}$ in $W^{\perp}$

Observe :

$$(y - \lambda_1 \underline{u}_1 - \lambda_2 \underline{u}_2 \ldots - \lambda_p \underline{u}_p) \cdot \underline{u}_i = 0 \iff y \cdot \underline{u}_i - \lambda_i \underline{u}_i \cdot \underline{u}_i = 0$$

$$\iff \lambda_i = \frac{y \cdot \underline{u}_i}{\underline{u}_i \cdot \underline{u}_i}$$

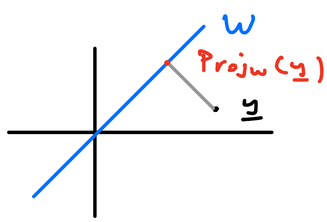
## Definition

If $W \subset \mathbb{R}^n$ is a subspace with orthogonal basis $\{\underline{u}_1, \dots, \underline{u}_p\}$, the <u>orthogonal projection of $\underline{y}$ onto $W$</u> is the vector

$$\mathrm{Proj}_W(\underline{y}) = \frac{\underline{y} \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \underline{u}_1 + \dots + \frac{\underline{y} \cdot \underline{u}_p}{\underline{u}_p \cdot \underline{u}_p} \underline{u}_p$$

$\hat{\underline{y}}$ in above notation.

Example in $\mathbb{R}^2$ :




## Important Facts

1/ Given $W \subset \mathbb{R}^n$ we can always find an orthogonal basis $\{\underline{u}_1, \dots, \underline{u}_p\}$ for $W$ (we'll see how next time).

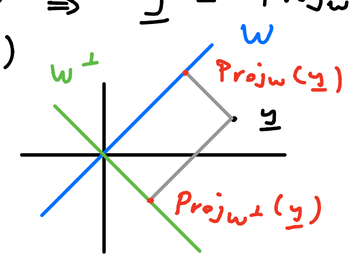
2/ $\mathrm{Proj}_W(\underline{y})$ does <u>not</u> depend on this choice of basis.

3/ There is <u>one and only one</u> way to write $\underline{y} = \underline{z}_1 + \underline{z}_2$ where $\underline{z}_1$ in $W$ and $\underline{z}_2$ in $W^\perp$

$\underline{z}_1 = \mathrm{Proj}_W(\underline{y})$

$\underline{z}_2 = \mathrm{Proj}_{W^\perp}(\underline{y})$ $\Rightarrow$ $\underline{y} = \mathrm{Proj}_W(\underline{y}) + \mathrm{Proj}_{W^\perp}(\underline{y})$

Example in $\mathbb{R}^2$ :

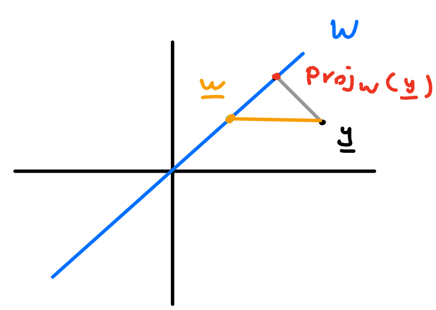



$\mathrm{Proj}_{W^\perp}(\underline{y})$

$\parallel$

3/ $\| \underline{y} - \mathrm{Proj}_W(\underline{y}) \| \leq \| \underline{y} - \underline{w} \|$ for all $\underline{w}$ in $W$, with equality $\Leftrightarrow$ $\underline{w} = \mathrm{Proj}_W(\underline{y})$

Example in $\mathbb{R}^2$ :



**Example** $\underline{y} = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $\underline{u}_1 = \begin{pmatrix}1\\-1\\1\end{pmatrix}$, $\underline{u}_2 = \begin{pmatrix}-1\\0\\1\end{pmatrix}$, $W = \mathrm{Span}(\underline{u}_1, \underline{u}_2)$

Find $\mathrm{Proj}_W(\underline{y})$. What is the minimum distance between $\underline{y}$ and $W$?

$$\mathrm{Proj}_W(\underline{y}) = \frac{\underline{y}\cdot\underline{u}_1}{\underline{u}_1\cdot\underline{u}_1}\underline{u}_1 + \frac{\underline{y}\cdot\underline{u}_2}{\underline{u}_2\cdot\underline{u}_2}\underline{u}_2 = \frac{2}{3}\begin{pmatrix}1\\-1\\1\end{pmatrix} + \frac{2}{2}\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}-1/3\\-2/3\\5/3\end{pmatrix}$$

Min distance between $\underline{y}$ and $W$

$$\underbrace{= \|\underline{y} - \mathrm{Proj}_W(\underline{y})\|}_{= \mathrm{Proj}_{W^\perp}(\underline{y})} = \left\|\begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}-1/3\\-2/3\\5/3\end{pmatrix}\right\|$$

$$= \left\|\begin{pmatrix}4/3\\8/3\\4/3\end{pmatrix}\right\| = \sqrt{\left(\tfrac{4}{3}\right)^2 + \left(\tfrac{8}{3}\right)^2 + \left(\tfrac{4}{3}\right)^2} = \frac{1}{3}\sqrt{96}$$

**Observation**:

If $\{\underline{u}_1, \dots, \underline{u}_p\}$ is an <u>orthonormal basis</u> of $W$ (ie $\underline{u}_i \cdot \underline{u}_i = 1$)

$$\Rightarrow \quad \mathrm{Proj}_W(\underline{y}) = (\underline{y}\cdot\underline{u}_1)\underline{u}_1 + \dots + (\underline{y}\cdot\underline{u}_p)\underline{u}_p.$$

$U = (\underline{u}_1 \dots \underline{u}_p)$ ← $n \times p$ matrix

$$\Rightarrow \quad \mathrm{Proj}_W(\underline{y}) = U\begin{pmatrix}\underline{u}_1\cdot\underline{y}\\ \vdots \\ \underline{u}_p\cdot\underline{y}\end{pmatrix} = U\begin{pmatrix}\underline{u}_1^T\underline{y}\\ \vdots \\ \underline{u}_p^T\underline{y}\end{pmatrix} = UU^T\underline{y}$$

**Conclusion**:

$\{\underline{u}_1, \dots, \underline{u}_p\}$ is an <u>orthonormal basis</u> of $W$ $\Rightarrow$ $\mathrm{Proj}_W(\underline{y}) = UU^T\underline{y}$

Sometimes useful

**Example** $W = \mathrm{Span}\Bigg(\underbrace{\begin{pmatrix}2/3\\1/3\\2/3\end{pmatrix}, \begin{pmatrix}-2/3\\2/3\\1/3\end{pmatrix}}_{\text{orthonormal basis}}\Bigg)$, $\underline{y} = \begin{pmatrix}4\\8\\1\end{pmatrix}$

$$U = \begin{pmatrix}2/3 & -2/3\\ 1/3 & 2/3\\ 2/3 & 1/3\end{pmatrix}, \quad U^T = \begin{pmatrix}2/3 & 1/3 & 2/3\\ -2/3 & 2/3 & 1/3\end{pmatrix} \Rightarrow UU^T = \begin{pmatrix}8/9 & -2/9 & 2/9\\ -2/9 & 5/9 & 4/9\\ 2/9 & 4/9 & 5/9\end{pmatrix}$$

$$\Rightarrow \quad \mathrm{Proj}_W(\underline{y}) = \begin{pmatrix} 8/9 & -2/9 & 2/9 \\ -2/9 & 5/9 & 4/9 \\ 2/9 & 4/9 & 5/9 \end{pmatrix} \begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$$

Direct Check : $\underline{y} \cdot \underline{u}_1 = 6$ , $\underline{y} \cdot \underline{u}_2 = 3$

$$\Rightarrow \quad \mathrm{Proj}_W(\underline{y}) = 6 \cdot \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix} + 3 \begin{pmatrix} -2/3 \\ 2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$$

Key Take Away : If $\{\underline{u}_1, \dots, \underline{u}_p\} \subset W$ is an orthogonal basis

orthogonal projection of $\underline{y}$ onto $W$

$$\mathrm{Proj}_W(\underline{y}) = \frac{\underline{y} \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \underline{u}_1 + \cdots + \frac{\underline{y} \cdot \underline{u}_p}{\underline{u}_p \cdot \underline{u}_p} \underline{u}_p$$

closest point to $\underline{y}$ in $W$

Next Task : Finding orthogonal bases.