Matrix Operations

Just special functions from $\mathbb{R}^n$ to $\mathbb{R}^m$

$$\{ m \times n \text{ matrices} \} = \{ \text{Linear transformations from } \mathbb{R}^n \text{ to } \mathbb{R}^m \}$$

Key observation : We can add, scale and compose functions so we can translate this to matrices.

Matrices	Linear Transformations
$A = (\underline{a}_1 \cdots \underline{a}_n)$ $m \times n$ matrices	$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$
$B = (\underline{b}_1 \cdots \underline{b}_n)$	$T_B : \mathbb{R}^n \longrightarrow \mathbb{R}^m$
$A + B := (\underline{a}_1 + \underline{b}_1 \; \underline{a}_2 + \underline{b}_2 \cdots \underline{a}_n + \underline{b}_n)$	$T_A + T_B = T_{A+B}$
term by term sum	sum of functions ie $(T_A + T_B)(\underline{x}) := T_A(\underline{x}) + T_B(\underline{x})$
$\lambda A := (\lambda \underline{a}_1 \; \lambda \underline{a}_2 \cdots \lambda \underline{a}_n)$	$\lambda T_A = T_{\lambda A}$
each entry multiplied by λ	T_A multiplied by constant λ ie $(\lambda T_A)(\underline{x}) := \lambda T_A(\underline{x})$

Example

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 4 \\ 3 & -1 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 5 & 9 \\ 5 & 3 & 12 \end{pmatrix}$$

$$2 \begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 8 & 14 \end{pmatrix}$$

Remark To sum matrices they must have same size.

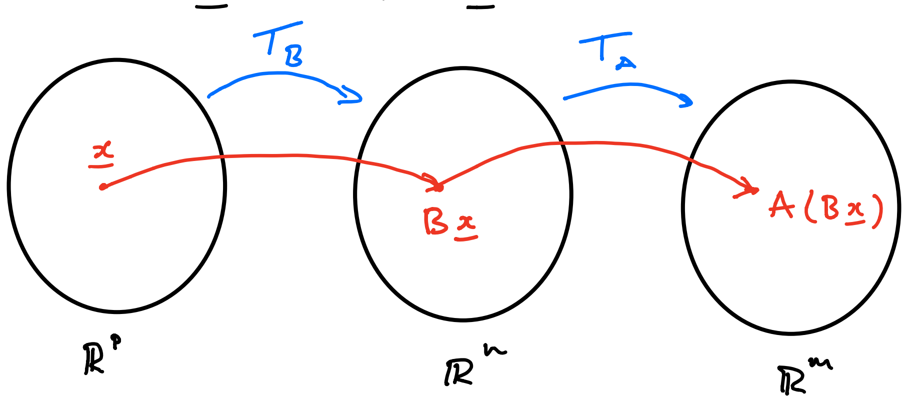
Composition is more complicated:

$$A = (\underline{a}_1 \cdots \underline{a}_n) \quad - \quad m \times n \text{ matrix}$$

$$B = (\underline{b}_1 \cdots \underline{b}_p) \quad - \quad n \times p \text{ matrix}$$

$$\Rightarrow \quad T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \quad \underline{v} \longmapsto A\underline{v}$$

$$T_B : \mathbb{R}^p \longrightarrow \mathbb{R}^n, \quad \underline{x} \longrightarrow B\underline{x}$$



Fact: $T_A \circ T_B : \mathbb{R}^p \to \mathbb{R}^m$ linear.

$$T_A \circ T_B(\underline{e}_1) = A(B\underline{e}_1) = A\underline{b}_1 \qquad \underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$T_A \circ T_B(\underline{e}_p) = A(B\underline{e}_p) = A\underline{b}_p \qquad \underline{e}_p = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow$ Standard Matrix of $T_A \circ T_B$ $= (A\underline{b}_1 \ A\underline{b}_2 \cdots A\underline{b}_p)$

Definition

$$\underset{m\times n}{A}\ \underset{n\times p}{B} := \overset{\text{matrix "multiplication"}}{} \underbrace{\left(A\underline{b}_1 \ A\underline{b}_2 \cdots A\underline{b}_p \right)}_{m\times p}$$

matrix "multiplication"

$m \times p$

($\underset{m\times n}{A}\ \underset{n\times 1\ (\text{vector in } \mathbb{R}^n)}{\underline{x}}$ can now be interpreted as matrix multiplication.)

Key Fact : Matrix Multiplication = Composition of linear transformations

i.e. $$T_{AB} = T_A \circ T_B$$

Row - Column Rule of Computing AB

$$\underset{ij^{th}\text{ entry of } AB}{(AB)_{ij}} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$\underbrace{\begin{array}{c|c} & \\ i^{th}\text{ row} \rightarrow & \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix} \end{array}}_{n} \qquad \begin{array}{c} j^{th}\text{ column} \\ \downarrow \\ \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ \vdots \\ b_{nj} \end{pmatrix} \end{array} \Big\} n$$

$$A \qquad\qquad\qquad B$$

Example

$$\underset{2\times3}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}} \underset{3\times2}{\begin{pmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}} = \begin{pmatrix} 1\cdot1 + 2\cdot0 \cdot 3\cdot1 & 1\cdot(-1) + 2\cdot2 + 3\cdot1 \\ 4\cdot1 + 5\cdot0 + 6\cdot1 & 4\cdot(-1) + 5\cdot2 + 6\cdot1 \end{pmatrix}$$

$$\underset{2\times 2}{\begin{pmatrix} 4 & 6 \\ 10 & 12 \end{pmatrix}}$$

Properties of Matrix Operations

1/ $A(BC) = (AB)C$

2/ $A(B+C) = AB + AC$

3/ $(A+B)C = AC + BC$

4/ $\lambda(AB) = (\lambda A)B = A(\lambda B)$

5/ $I_m A = A = A I_n$

Warning In general $AB \neq BA$. Also

zero matrix

$AB = 0 \not\Rightarrow A = 0$ or $B = 0$. This means

$AC = AB \not\Rightarrow C = B$ in general. E.g. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Definition $A = m \times n$ matrix

$A^T = A$ "transpose"

$(A^T)_{ij} := A_{ji}$

Example $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$

Facts 1/ $(A^T)^T = A$

2/ $(A+B)^T = A^T + B^T$

3/ $(rA)^T = r A^T$

4/ $(AB)^T = B^T A^T$