

The Matrix of an Abstract Linear Transformation.

Aim : Use coordinate systems to think about linear

$T: V \rightarrow W$ more concretely.

$\mathcal{B} = \{\underline{b}_1, \underline{b}_2, \dots, \underline{b}_n\}$ a basis for V . $\dim(V) = n$

$\mathcal{C} = \{\underline{c}_1, \underline{c}_2, \dots, \underline{c}_m\}$ a basis for W . $\Rightarrow \dim(W) = m$

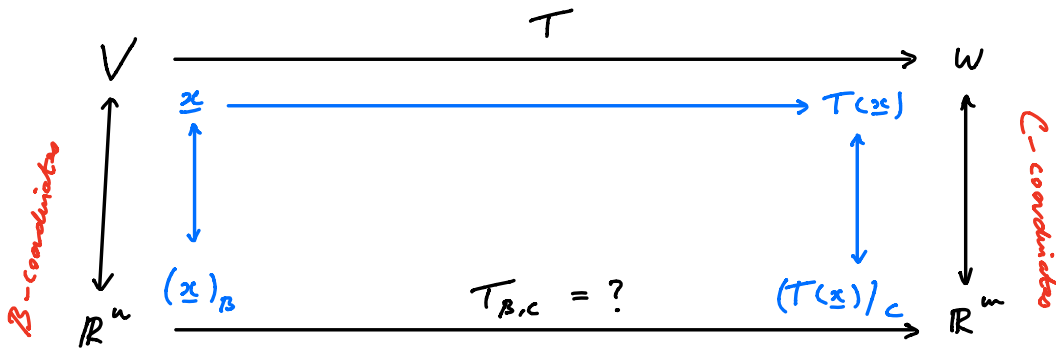
Recall :

$V \longleftrightarrow \mathbb{R}^n$ ↙ one-to-one, onto, linear ↘
 \mathcal{B} -coordinates

$\underline{x} \longleftrightarrow (\underline{x})_{\mathcal{B}} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}, \quad \underline{x} = \lambda_1 \underline{b}_1 + \dots + \lambda_n \underline{b}_n$

$W \longleftrightarrow \mathbb{R}^m$ ↙ ↘
 \mathcal{C} -coordinates

$\underline{v} \longleftrightarrow (\underline{v})_{\mathcal{C}} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix}, \quad \underline{v} = \mu_1 \underline{c}_1 + \dots + \mu_m \underline{c}_m$



Fact $T_{\mathcal{B}, \mathcal{C}}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear, so there exists a matrix $A_{\mathcal{B}, \mathcal{C}}$ such that $A_{\mathcal{B}, \mathcal{C}} (\underline{x})_{\mathcal{B}} = (T(\underline{x}))_{\mathcal{C}}$.

Recall, $(\underline{b}_i)_{\mathcal{B}} = \underline{e}_i$

\Rightarrow i^{th} column of $A_{\mathcal{B}, \mathcal{C}} = A_{\mathcal{B}, \mathcal{C}} \underline{e}_i = A_{\mathcal{B}, \mathcal{C}} (\underline{b}_i)_{\mathcal{B}} = (T(\underline{b}_i))_{\mathcal{C}}$

Definition The matrix of $T: V \rightarrow W$ with respect to bases \mathcal{B} and \mathcal{C} is $A_{\mathcal{B}, \mathcal{C}} = ((T(\underline{b}_1))_{\mathcal{C}} \dots (T(\underline{b}_n))_{\mathcal{C}})$

Key Property : $A_{\mathcal{B}, \mathcal{C}} (\underline{x})_{\mathcal{B}} = (T(\underline{x}))_{\mathcal{C}}$ for all \underline{x} in V

Example

$E_n = \text{standard basis of } \mathbb{R}^n$

$E_m = \text{standard basis of } \mathbb{R}^m$

$$\text{If } T: \mathbb{R}^n \rightarrow \mathbb{R}^m \quad \Rightarrow \quad A_{E_n, E_m} = A \\ \underline{x} \rightarrow A\underline{x}$$

Example $V = W = \mathbb{P}_2(\mathbb{R}), B = C = \{1, x, x^2\}$

β-coordinates

$$T: \mathbb{P}_2(\mathbb{R}) \rightarrow \mathbb{P}^2(\mathbb{R}) \quad \leftarrow \text{derivative} \quad a_0 + a_1x + a_2x^2 \longleftrightarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \\ p(x) \mapsto p'(x)$$

$$A_{B,B} = ((T(1))_B \ (T(x))_B \ (T(x^2))_B) \\ = ((0)_B \ (1)_B \ (2x)_B) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Example $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$
 $\underline{x} \rightarrow \begin{pmatrix} -1 & 3 \\ -3/2 & 7/2 \end{pmatrix} \underline{x}, B = C = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

$$A_{B,B} = (T\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)_B \ T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)_B) \\ = \left(\left(\frac{1}{2} \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)_B \ (2 \begin{pmatrix} 1 \\ 1 \end{pmatrix})_B \right) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$$

General Case : $V = W = \mathbb{R}^n, A - n \times n \text{ matrix}, B \text{ basis.}$

$$A_{B,B} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \Leftrightarrow B \text{ is a basis of eigenvectors of } A$$

Q, If $V = \mathbb{R}^n, W = \mathbb{R}^m, T = T_A$ for A $n \times n$ matrix is there
more direct way to find $A_{B,C}$?

Recall:

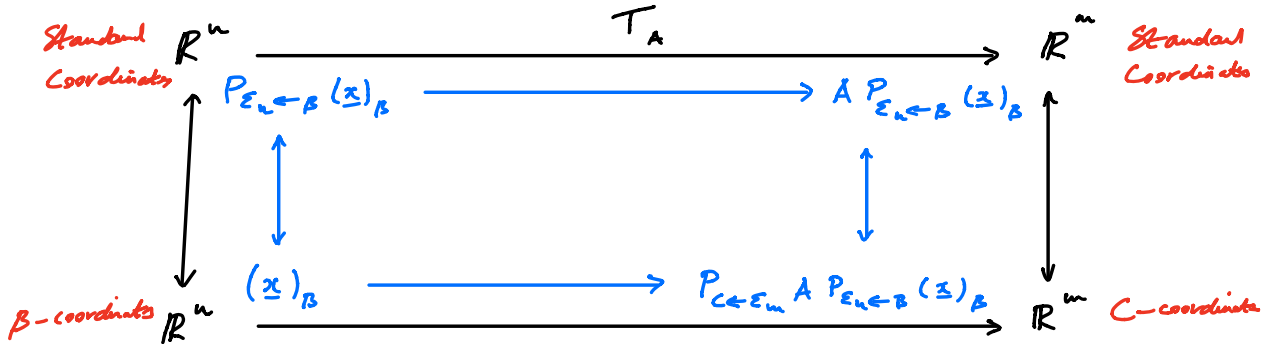
$$\gamma \ (\underline{v})_{E_n} = \underline{v} \quad \leftarrow \text{standard coordinates are ordered coordinates}$$

$$2/ P_{\mathcal{E}_n \leftarrow \mathcal{B}} = ((\underline{b}_1)_{\mathcal{E}_n} \dots (\underline{b}_n)_{\mathcal{E}_n}) = (\underline{b}_1 \dots \underline{b}_n)$$

$$\text{and } P_{\mathcal{E}_n \leftarrow \mathcal{B}} (\underline{x})_{\mathcal{B}} = (\underline{x})_{\mathcal{E}_n} = \underline{x} \text{ for all } \underline{x} \text{ in } \mathbb{R}^n$$

$$3/ P_{\mathcal{E}_n \leftarrow \mathcal{B}} (\underline{x})_{\mathcal{B}} = (\underline{x})_{\mathcal{E}_n} \Rightarrow (P_{\mathcal{E}_n \leftarrow \mathcal{B}})^{-1} (\underline{x})_{\mathcal{E}_n} = (\underline{x})_{\mathcal{B}}$$

$$\Rightarrow (P_{\mathcal{E}_n \leftarrow \mathcal{B}})^{-1} = P_{\mathcal{B} \leftarrow \mathcal{E}_n}$$



Conclusion

$$A_{\mathcal{B}, \mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{E}_m} A P_{\mathcal{E}_n \leftarrow \mathcal{B}}$$

$$= (P_{\mathcal{E}_m \leftarrow \mathcal{C}})^{-1} A P_{\mathcal{E}_n \leftarrow \mathcal{B}}$$

$$= (\underline{c}_1 \dots \underline{c}_m)^{-1} A (\underline{b}_1 \dots \underline{b}_n)$$

Switch back from standard to C-coordinates (pointing to $P_{\mathcal{C} \leftarrow \mathcal{E}_m}$)
Apply T_A (pointing to A)
switch from B to standard coordinates (pointing to $P_{\mathcal{E}_n \leftarrow \mathcal{B}}$)

Important Example A - $n \times n$ matrix, $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$

basis of eigenvectors, $A \underline{b}_i = \lambda_i \underline{b}_i$, $P = (\underline{b}_1 \dots \underline{b}_n)$

$$\Rightarrow P^{-1} A P = A_{\mathcal{B}, \mathcal{B}} = \begin{pmatrix} \lambda_1 & & 0 \\ & \dots & \\ 0 & & \lambda_n \end{pmatrix}$$

Important Future Goal: Find basis \mathcal{B}, \mathcal{C} to make $A_{\mathcal{B}, \mathcal{C}}$

as simple as possible.