The Matrix of an Abstract Linear Transformation.

Aim: Use coordinate systems to think about linear $T : V \to W$ more concretely.

$\mathcal{B} = \{\underline{b}_1, \underline{b}_2, \dots, \underline{b}_n\}$ a basis for V.
$\mathcal{C} = \{\underline{c}_1, \underline{c}_2, \dots, \underline{c}_m\}$ a basis for W.

$\Rightarrow$ $\dim(V) = n$, $\dim(W) = n$

Recall:

one-to-one, onto, linear

$V \longleftrightarrow \mathbb{R}^n$

$\mathcal{B}$-coordinates

$\underline{x} \longleftrightarrow (\underline{x})_{\mathcal{B}} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$, $\quad \underline{x} = \lambda_1 \underline{b}_1 + \dots + \lambda_n \underline{b}_n$

$W \longleftrightarrow \mathbb{R}^m$

$\mathcal{C}$ - coordinates

$\underline{v} \longleftrightarrow (\underline{v})_{\mathcal{C}} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix}$, $\quad \underline{v} = \mu_1 \underline{c}_1 + \dots + \mu_m \underline{c}_m$

$$\begin{tikzcd}
V \arrow[rrr, "T"] \arrow[ddd, leftrightarrow, "\mathcal{B}\text{-coordinates}"'] & & & W \arrow[ddd, leftrightarrow, "\mathcal{C}\text{-coordinates}"] \\
& \underline{x} \arrow[r] \arrow[d, leftrightarrow] & T(\underline{x}) \arrow[d, leftrightarrow] & \\
& (\underline{x})_{\mathcal{B}} & (T(\underline{x}))_{\mathcal{C}} & \\
\mathbb{R}^n \arrow[rrr, "T_{\mathcal{B},\mathcal{C}} = ?"] & & & \mathbb{R}^m
\end{tikzcd}$$

Fact $T_{\mathcal{B},\mathcal{C}} : \mathbb{R}^n \to \mathbb{R}^m$ linear, so there exists a matrix $A_{\mathcal{B},\mathcal{C}}$ such that $A_{\mathcal{B},\mathcal{C}} (\underline{x})_{\mathcal{B}} = (T(\underline{x}))_{\mathcal{C}}$.

Recall, $(\underline{b}_i)_{\mathcal{B}} = \underline{e}_i$

$\Rightarrow$ i^{th} column of $A_{\mathcal{B},\mathcal{C}} = A_{\mathcal{B},\mathcal{C}} \underline{e}_i = A_{\mathcal{B},\mathcal{C}} (\underline{b}_i)_{\mathcal{B}} = (T(\underline{b}_i))_{\mathcal{C}}$

Definition The matrix of $T : V \to W$ with respect to bases $\mathcal{B}$ and $\mathcal{C}$ is $A_{\mathcal{B},\mathcal{C}} = \left((T(b_1))_{\mathcal{C}} \dots T(\underline{b}_n)_{\mathcal{C}} \right)$

Key Property: $A_{\mathcal{B},\mathcal{C}} (\underline{x})_{\mathcal{B}} = (T(\underline{x}))_{\mathcal{C}}$ for all $\underline{x}$ in V

Example

$\mathcal{E}_n$ = standard basis of $\mathbb{R}^n$

$\mathcal{E}_m$ = standard basis of $\mathbb{R}^m$

If $T: \mathbb{R}^n \to \mathbb{R}^m$, $\underline{x} \longrightarrow A\underline{x}$ $\Rightarrow$ $A_{\mathcal{E}_n, \mathcal{E}_m} = A$

Example $V = W = \mathbb{P}_2(\mathbb{R})$, $\mathcal{B} = \mathcal{C} = \{1, x, x^2\}$

$T: \mathbb{P}_2(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$, $p(x) \longmapsto p'(x)$ ← derivative

$\mathcal{B}$-coordinates

$a_0 + a_1 x + a_2 x^2 \longleftrightarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$

$$A_{\mathcal{B},\mathcal{B}} = \left((T(1))_{\mathcal{B}} \ (T(x))_{\mathcal{B}} \ (T(x^2))_{\mathcal{B}} \right)$$
$$= \left((0)_{\mathcal{B}} \ (1)_{\mathcal{B}} \ (2x)_{\mathcal{B}} \right) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Example $T: \mathbb{R}^2 \to \mathbb{R}^2$, $\underline{x} \longrightarrow \begin{pmatrix} -1 & 3 \\ -3/2 & 7/2 \end{pmatrix} \underline{x}$, $\mathcal{B} = \mathcal{C} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

$$A_{\mathcal{B},\mathcal{B}} = \left(\left(T\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)_{\mathcal{B}} \ \left(T\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)_{\mathcal{B}} \right)$$
$$= \left(\left(\tfrac{1}{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)_{\mathcal{B}} \ \left(2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)_{\mathcal{B}} \right) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$$

General Case : $V = W = \mathbb{R}^n$, A - $n \times n$ matrix, $\mathcal{B}$ basis.

$A_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \iff \mathcal{B}$ is a basis of eigenvectors of A

Q/ If $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, $T = T_A$ for A $m \times n$ matrix is there more direct way to find $A_{\mathcal{B},\mathcal{C}}$?

Recall :

1/ $(\underline{v})_{\mathcal{E}_n} = \underline{v}$ ← standard coordinates are ordinary coordinates

2/ $P_{\mathcal{E}_n \leftarrow \mathcal{B}} = \left((\underline{b}_1)_{\mathcal{E}_n} \cdots (\underline{b}_n)_{\mathcal{E}_n} \right) = (\underline{b}_1 \dots \underline{b}_n)$

and $P_{\mathcal{E}_n \leftarrow \mathcal{B}} (\underline{x})_{\mathcal{B}} = (\underline{x})_{\mathcal{E}_n} = \underline{x}$ for all $\underline{x}$ in $\mathbb{R}^n$

3/ $P_{\mathcal{E}_n \leftarrow \mathcal{B}} (\underline{x})_{\mathcal{B}} = (\underline{x})_{\mathcal{E}_n} \Rightarrow (P_{\mathcal{E}_n \leftarrow \mathcal{B}})^{-1} (\underline{x})_{\mathcal{E}_n} = (\underline{x})_{\mathcal{B}}$

$\Rightarrow \quad (P_{\mathcal{E}_n \leftarrow \mathcal{B}})^{-1} = P_{\mathcal{B} \leftarrow \mathcal{E}_n}$

$$
\begin{tikzcd}
\text{Standard Coordinates} & \mathbb{R}^n \arrow[rrr, "T_A"] \arrow[dd, leftrightarrow] & & & \mathbb{R}^m \arrow[dd, leftrightarrow] & \text{Standard Coordinates} \\
& & P_{\mathcal{E}_n \leftarrow \mathcal{B}} (\underline{x})_{\mathcal{B}} \arrow[r] \arrow[d, leftrightarrow] & A P_{\mathcal{E}_n \leftarrow \mathcal{B}} (\underline{x})_{\mathcal{B}} \arrow[d, leftrightarrow] & & \\
\mathcal{B}\text{-coordinates} & \mathbb{R}^n \arrow[rrr] & (\underline{x})_{\mathcal{B}} \arrow[r] & P_{\mathcal{C} \leftarrow \mathcal{E}_m} A P_{\mathcal{E}_n \leftarrow \mathcal{B}} (\underline{x})_{\mathcal{B}} & \mathbb{R}^m & \mathcal{C}\text{-coordinates}
\end{tikzcd}
$$

<u>Conclusion</u>

Switch back from standard to $\mathcal{C}$-coordinates   Apply T_A switch from $\mathcal{B}$ to standard coordinates

$$
\begin{aligned}
A_{\mathcal{B},\mathcal{C}} &= P_{\mathcal{C} \leftarrow \mathcal{E}_m} \; A \; P_{\mathcal{E}_n \leftarrow \mathcal{B}} \\
&= (P_{\mathcal{E}_m \leftarrow \mathcal{C}})^{-1} A \, P_{\mathcal{E}_n \leftarrow \mathcal{B}} \\
&= (\underline{c}_1 \dots \underline{c}_n)^{-1} A \, (\underline{b}_1 \dots \underline{b}_n)
\end{aligned}
$$

<u>Important Example</u> $\quad A$ - $n \times n$ matrix, $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$

basis of eigenvectors, $A\underline{b}_i = \lambda_i \underline{b}_i$, $P = (\underline{b}_1 \dots \underline{b}_n)$

$$\Rightarrow \quad P^{-1} A P = A_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

<u>Important Future Goal</u>: Find bases $\mathcal{B}, \mathcal{C}$ to make $A_{\mathcal{B},\mathcal{C}}$ as simple as possible.