Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

$$\begin{tikzcd}
 & A \text{ - } m\times n \text{ matrix} \arrow[dl, "\substack{\text{Solutions to} \\ \text{linear system} \\ \text{with coefficient} \\ \text{matrix } A}"'] \arrow[dr, "\substack{\text{Columns of } A \\ \underline{a}_1,\, \underline{a}_2, \dots,\, \underline{a}_n}"] & \\
\mathbb{R}^n \arrow[rr, "T_A"] & & \mathbb{R}^m
\end{tikzcd}$$

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longrightarrow x_1\underline{a}_1 + x_2\underline{a}_2 + \dots + x_n\underline{a}_n = A\underline{x}$$

Conclusion

Given A, an $m\times n$ matrix, we can construct a function

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \qquad \underline{x} \longmapsto A\underline{x}.$$

We call T_A the linear transformation associated to A.

Example $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$

$$\Rightarrow \quad T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto x_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2\begin{pmatrix} 3 \\ 4 \end{pmatrix} + x_3\begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

E.g. $T_A\left(\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \end{pmatrix} + 3\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 13 \\ 16 \end{pmatrix}$

Properties of T_A: 1) $A(\underline{u}+\underline{v}) = A\underline{u} + A\underline{v}$ for all $\underline{u}, \underline{v}$ in $\mathbb{R}^n$

$$\Rightarrow \quad T_A(\underbrace{\underline{u}+\underline{v}}_{\text{addition in } \mathbb{R}^n}) = \underbrace{T_A(\underline{u}) + T_A(\underline{v})}_{\text{addition in } \mathbb{R}^m}$$

2/ $A(\lambda \underline{u}) = \lambda(A\underline{u})$ for all $\underline{u}$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$

$$\Rightarrow \underbrace{T_A(\lambda \underline{u})}_{\text{Scalar Multiplication in } \mathbb{R}^n} = \underbrace{\lambda T_A(\underline{u})}_{\text{Scalar Multiplication in } \mathbb{R}^m}$$

$\Rightarrow T_A : \mathbb{R}^n \to \mathbb{R}^m$ preserves addition and scalar multiplication of vectors.

This is <u>very special.</u> Most functions don't do this.

E.g. $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1^2 + 1 \\ x_1 + x_2 \end{pmatrix}$$

$$\Rightarrow f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = f\left(\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 9+1 \\ 3+2 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$$

$$\neq$$

$$f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) + f\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1+1 \\ 1+1 \end{pmatrix} + \begin{pmatrix} 4+1 \\ 2+1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

<u>Definition</u> We say a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a <u>linear transformation</u> if

$$\left.\begin{aligned} &1/\quad T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v}) \\ &2/\quad T(\lambda \underline{u}) = \lambda T(\underline{u}) \end{aligned}\right\} \text{ for all } \underline{u}, \underline{v} \text{ in } \mathbb{R}^n \text{ and } \lambda \text{ in } \mathbb{R}$$

A – $m \times n$ matrix $\Rightarrow$ T_A Linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

Q/ Given $T: \mathbb{R}^n \to \mathbb{R}^m$ linear, can we find a matrix A such that $T = T_A$?

<u>Definition</u> $\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, ..., $\underline{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

Note that $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\underline{e}_1 + x_2\underline{e}_2 + \cdots + x_n\underline{e}_n$

$$\Rightarrow T(\underline{x}) = T(x_1\underline{e}_1 + \cdots + x_n\underline{e}_n)$$

vector in $\mathbb{R}^m$

$$\overset{\text{Linear}}{=} x_1 T(\underline{e}_1) + x_2 T(\underline{e}_2) + \cdots + x_n T(\underline{e}_n)$$

$$= \underbrace{\big(T(\underline{e}_1)\ T(\underline{e}_2)\ \cdots\ T(\underline{e}_n)\big)}_{m \times n \text{ matrix}} \underline{x}$$

<u>Definition</u> Given $T: \mathbb{R}^n \to \mathbb{R}^m$ linear, the <u>standard matrix</u> associated to T is the $m \times n$ matrix

$$A_T = \big(T(\underline{e}_1)\ T(\underline{e}_2)\ \cdots\ T(\underline{e}_n)\big)$$

<u>Key Fact</u> : $T(\underline{x}) = A_T \underline{x}$ for all $\underline{x}$ in $\mathbb{R}^n$

Conclusion :

$$\{m \times n \text{ matrices}\} = \left\{\begin{array}{l}\text{Linear Transformations} \\ \text{from } \mathbb{R}^n \text{ to } \mathbb{R}^m.\end{array}\right\}$$

$$A \longmapsto T_A$$

$$A_T \longleftarrow\!\shortmid\ T$$

Crucial Observation : If $T : \mathbb{R}^n \to \mathbb{R}^n$ linear then T is completely determined by $T(\underline{e}_1), \dots, T(\underline{e}_n)$.

Important Examples : 1) $Id : \mathbb{R}^n \to \mathbb{R}^n$
$\underline{x} \mapsto \underline{x}$

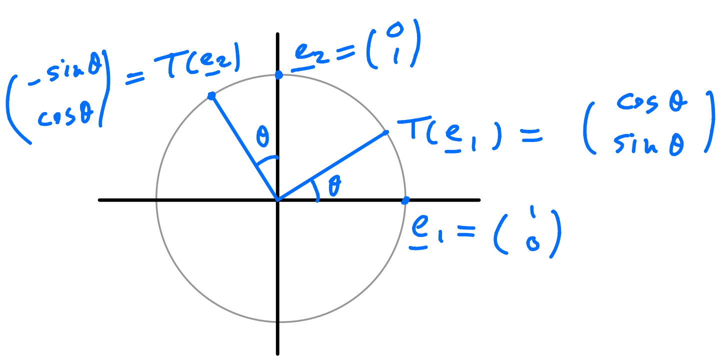
$$A_{Id} = \left(Id(\underline{e}_1) \cdots Id(\underline{e}_n) \right) = \left(\underline{e}_1 \cdots \underline{e}_n \right)$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1 \ddots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & 1 \end{pmatrix} = I_n$$

called the $n \times n$ identity matrix.

By construction $I_n \underline{x} = \underline{x}$ for all $\underline{x}$ in $\mathbb{R}^n$

2) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ rotation about $\underline{0}$ by angle θ counterclockwise is linear.



standard matrix associated to T

$\Rightarrow$ $T(\underline{x}) = \underline{x}$ rotated by θ counterclockwise around $\underline{0}$ $= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \underline{x}$