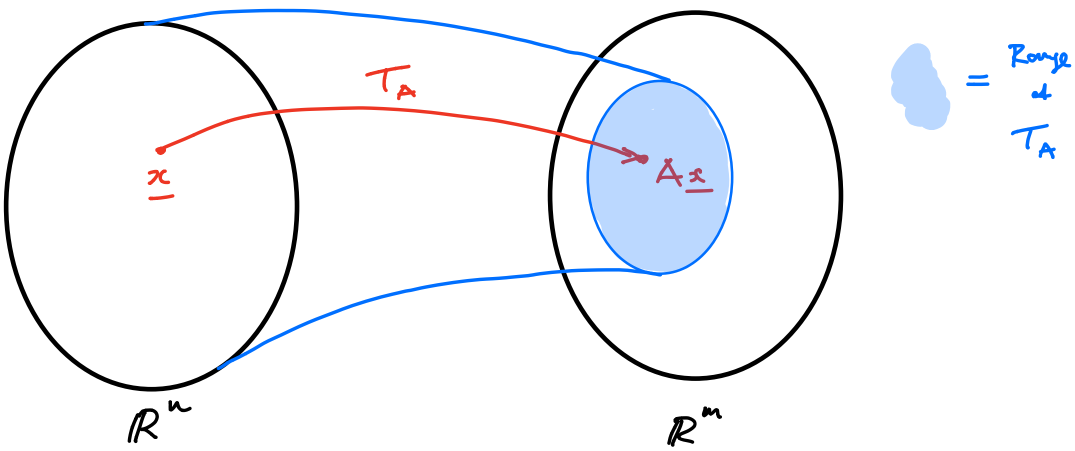
Linear Transformations and Linear System Solutions

A – $m\times n$ matrix $\rightsquigarrow$ $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ Linear transformation

$\underline{x} \longmapsto A\underline{x}$

Range of T_A := Image of $\mathbb{R}^n$ under T_A $= \text{Span}(\underline{a}_1 \ldots \underline{a}_n)$



$\underline{b}$ in Range of T_A $\Longleftrightarrow$ $\underline{b}$ in $\text{Span}(\underline{a}_1, \ldots, \underline{a}_n)$ $\Longleftrightarrow$ $(A \mid \underline{b})$ consistent

<u>Definition</u> : T_A <u>onto</u> $\Longleftrightarrow$ Range of $T_A = \mathbb{R}^m$

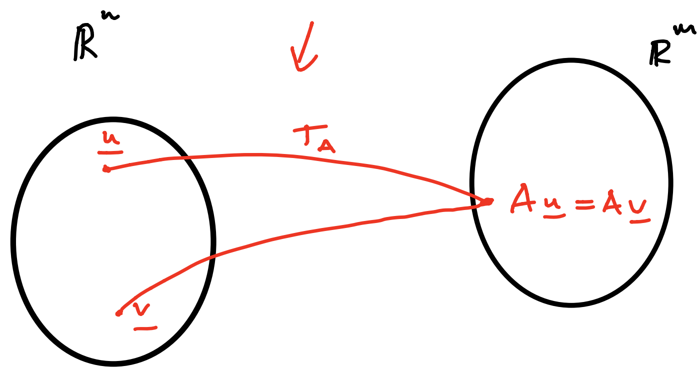
T_A <u>onto</u> $\Longleftrightarrow$ $\text{Span}(\underline{a}_1, \ldots, \underline{a}_n) = \mathbb{R}^m$ $\Longleftrightarrow$ $(A \mid \underline{b})$ consistent for any $\underline{b}$.

$\Updownarrow$

There is a pivot position in <u>every row</u> of reduced A.

Definition T_A one-to-one $\iff$ $T_A(\underline{u}) = T_A(\underline{v}) \Rightarrow \underline{u} = \underline{v}$

Not one-to-one



T_A one-to-one $\iff$ $(A \mid \underline{b})$ consistent admits a unique solution $\iff$ $\{\underline{a}_1, \ldots, \underline{a}_n\}$ linearly independent

$\Updownarrow$

Reduced A has pivot position in every column

Example $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$ Is T_A onto? Is T_A one-to-one?

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \longmapsto \begin{pmatrix} \boxed{1} & 3 & 5 \\ 0 & \boxed{-2} & -4 \end{pmatrix}$$

$\Rightarrow$ 1/ Every row has pivot position $\Rightarrow$ T_A onto

2/ Not every column has pivot position $\Rightarrow$ T_A not one-to-one.