Inverse of a Matrix

A – $n \times n$ matrix

Definition A is invertible if there exist B an $n \times n$ matrix such that $AB = BA = I_n$.

Fact: Such a B is unique and we write $B = A^{-1}$
if it exists — inverse of A.

Example $\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Important Observation:

identity function

$$T_A \circ T_{A^{-1}} = T_{AA^{-1}} = T_{I_n} = Id_{\mathbb{R}^n}$$

$$T_{A^{-1}} \circ T_A = T_{A^{-1}A} = T_{I_n} = Id_{\mathbb{R}^n}$$

$\Rightarrow$ $T_{A^{-1}}$ = inverse function of T_A.

Recall: A function has an inverse if and only if it is both one-to-one and onto.

Conclusion:

A invertible ($n \times n$) $\iff$ $T_A : \mathbb{R}^n \to \mathbb{R}^n$ one-to-one and onto $\iff$ $\mathrm{Span}(\underline{a}_1, \dots, \underline{a}_n) = \mathbb{R}^n$ and $\{\underline{a}_1, \dots, \underline{a}_n\}$ linearly independent

Interesting coincidence in $n \times n$ case:

$\text{Span}(\underline{a}_1, \dots, \underline{a}_n) = \mathbb{R}^n$ $\quad\Updownarrow\quad$ Reduced A has pivot position in <u>every row</u>

$\{\underline{a}_1, \dots, \underline{a}_n\}$ Linearly indepent $\quad\Updownarrow\quad$ Reduced A has pivot in <u>every column</u>

Reduced A has pivot position in <u>every row</u> $\Leftrightarrow$ Reduced A has pivot in <u>every column</u>

must be $n \times n$

E.g.

For $n = 3$ only possible echelon form is $\begin{pmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{pmatrix}$

$\Rightarrow$ For A $n \times n$ matrix T_A <u>onto</u> $\Leftrightarrow$ T_A one-to-one.

A $n \times n$ matrix <u>Invertible</u>

$\Updownarrow$

$T_A : \mathbb{R}^n \to \mathbb{R}^n$ one-to-one $\quad\Updownarrow\quad$ $\{\underline{a}_1, \dots, \underline{a}_n\}$ Linearly independent $\quad\Updownarrow\quad$ Reduced A has a pivot in <u>every column</u>

$T_A : \mathbb{R}^n \to \mathbb{R}^n$ onto $\quad\Updownarrow\quad$ $\text{Span}(\underline{a}_1, \dots, \underline{a}_n) = \mathbb{R}^n$ $\quad\Updownarrow\quad$ Reduced A has pivot in <u>every row</u>

$\Updownarrow$

A is row equivalent to I_n

Computing A^{-1}

<u>$n=2$</u> $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Called determinant of A

A invertible $\iff$ $\det(A) := ad-bc \neq 0$

In this case $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

<u>Example</u> $\quad \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}^{-1} = \frac{1}{3\cdot 1 - 2\cdot 1}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$

For $n>2$ general form is much more complicated.

Recall A invertible $\iff$ A row equivalent to I_n

Fact : $(A \mid I_n)$ is row equivalent to $(I_n \mid A^{-1})$

reduced echelon form

Algorithm to find A^{-1} (if it exists) :

1/ Write $(A \mid I_n)$

2/ Put in reduced echelon form $(I_n \mid B)$

3/ $B = A^{-1}$.

<u>Example</u> Is $\begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ invertible? If so compute its inverse.

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -5 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 6 & -2 & 1 & 5 \end{array}\right) \longleftarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -5 & 1 & -2 & 1 & 0 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{-1}{3} & 1/6 & 5/6 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/3 & -1/6 & 1/6 \\ 0 & 0 & 1 & \frac{-1}{3} & 1/6 & 5/6 \end{array}\right)$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1/2 & -1/2 \\ 1/3 & -1/6 & 1/6 \\ \frac{-1}{3} & 1/6 & 5/6 \end{pmatrix} \Longleftarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1/2 & -1/2 \\ 0 & 1 & 0 & 1/3 & -1/6 & 1/6 \\ 0 & 0 & 1 & \frac{-1}{3} & 1/6 & 5/6 \end{array}\right)$$