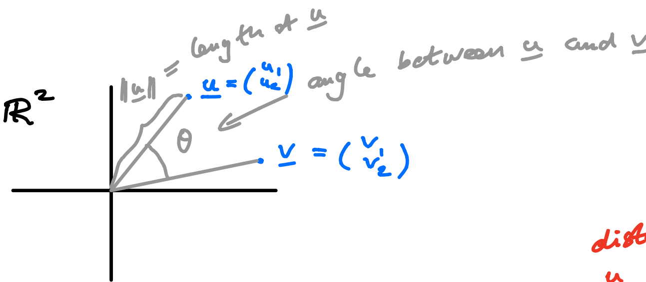
Inner Products, Lengths and Orthogonality

Aim : Introduce concepts of lengths and angles into theory of vector spaces.

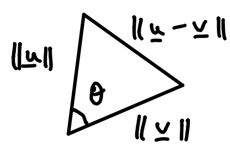
Example : $V = \mathbb{R}^2$



distance between $\underline{u}$ and $\underline{v}$

Pythagoras : $\|u\| = \sqrt{u_1^2 + u_2^2}$, $\|\underline{v}\| = \sqrt{v_1^2 + v_2^2}$, $\|\underline{u} - \underline{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$

Trigonometry :
(Law of Cosines)



$\Rightarrow \quad \|\underline{u} - \underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2 - 2\|\underline{u}\|\|\underline{v}\|\cos\theta$

$\Rightarrow \quad u_1 v_1 + u_2 v_2 = \|\underline{u}\| . \|\underline{v}\| \cos\theta$

important quantity

The same basic formulae hold in $\mathbb{R}^3$.

Definition Given $\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, $\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ in $\mathbb{R}^n$ the **scalar product** (or dot product, or standard inner product) of $\underline{u}$ and $\underline{v}$ is the number

just notation

$\underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n .$

transpose of A

Recall : A – $m \times n$ matrix $\Rightarrow$ A^T – $n \times m$ matrix and $(A^T)_{ij} = (A)_{ji}$

$\underline{u}$ = $n \times 1$ matrix $\Rightarrow$ $\underline{u}^T$ = $1 \times n$ matrix

$\underline{u} \cdot \underline{v} = \underline{u}^T \underline{v} = (u_1 \dots u_n) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1 v_1 + \dots + u_n v_n$

$\mathbb{R}^2 / \mathbb{R}^3$: 1/ Length of $\underline{u} = \sqrt{\underline{u} \cdot \underline{u}}$

2/ $\underline{u} \cdot \underline{v} = \|\underline{u}\| \cdot \|\underline{v}\| \cos(\theta)$

Properties of standard inner product on $\mathbb{R}^n$:

1/ $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$

2/ $(\underline{u} + \underline{v}) \cdot \underline{w} = \underline{u} \cdot \underline{w} + \underline{v} \cdot \underline{w}$

3/ $(\lambda \underline{u}) \cdot \underline{v} = \lambda (\underline{u} \cdot \underline{v})$

4/ $\underline{u} \cdot \underline{u} \geq 0$ and $\underline{u} \cdot \underline{u} = 0 \iff \underline{u} = \underline{0}$

Definitions

Length / Norm :

Given $\underline{u}$ in $\mathbb{R}^n$

$\|\underline{u}\| := \sqrt{\underline{u} \cdot \underline{u}}$ = length (or norm) of $\underline{u}$

$\|\underline{u}\| = 1 \implies \underline{u}$ is a unit vector

absolute value of λ — Called the normalization of $\underline{u}$

Fact: $\|\lambda \underline{u}\| = |\lambda| \cdot \|\underline{u}\| \implies \frac{1}{\|\underline{u}\|} \underline{u}$ is unit vector

Example $\underline{u} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix} \implies \|\underline{u}\| = \sqrt{1^2 + 2^2 + 0^2 + (-1)^2} = \sqrt{6}$

$\implies \frac{1}{\sqrt{6}} \underline{u} = \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 0 \\ -1/\sqrt{6} \end{pmatrix}$ = unit vector

Distance

Given $\underline{u}, \underline{v}$ in $\mathbb{R}^n$

$\text{dist}(\underline{u}, \underline{v}) := \|\underline{u} - \underline{v}\|$ = distance between $\underline{u}$ and $\underline{v}$.

Example $\underline{u} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}, \underline{v} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 3 \end{pmatrix} \implies \|\underline{u} - \underline{v}\| = \sqrt{(1-(-1))^2 + 2^2 + 0^2 + (-1-3)^2}$

$= \sqrt{24}$

Orthogonality

Given $\underline{u}, \underline{v}$ in $\mathbb{R}^n$

$\underline{u}, \underline{v}$ are orthogonal $\iff \underline{u} \cdot \underline{v} = 0$

$\mathbb{R}^2 / \mathbb{R}^3$: $\underline{u}, \underline{v}$ orthogonal $\Leftrightarrow$ $\cos\theta = 0 \Leftrightarrow \theta = \frac{\pi}{2}$ ($\underline{u}, \underline{v}$ at right angles)

Useful Property :

$$\|\underline{u}+\underline{v}\|^2 = (\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v}) = \underline{u}\cdot\underline{u} + \underline{u}\cdot\underline{v} + \underline{v}\cdot\underline{u} + \underline{v}\cdot\underline{v}$$
$$= \|\underline{u}\|^2 + 2\,\underline{u}\cdot\underline{v} + \|\underline{v}\|^2$$

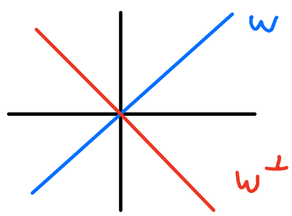
$\underline{u}, \underline{v}$ orthogonal $\Rightarrow$ $\|\underline{u}+\underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2$ (This is Pythagoras' Theorem)

Given $W \subset \mathbb{R}^n$ a subspace

Called the orthogonal complement of W.

$W^\perp = \{ \underline{u}$ in $\mathbb{R}^n$ such that $\underline{u}\cdot\underline{w} = 0$ for all $\underline{w}$ in $W \}$

Example



Properties of $W^\perp$

1/ $W^\perp \subset \mathbb{R}^n$ is a subspace

2/ If $W = \mathrm{Span}(\underline{v}_1, \dots, \underline{v}_p)$, $\underline{u}$ in $W^\perp \Leftrightarrow \underline{u}\cdot\underline{v}_i = 0$ for all i.

3/ $(W^\perp)^\perp = W$

Theorem Given A – $m \times n$ matrix then

$$(\mathrm{Col}(A))^\perp = \mathrm{Nul}(A^T) \quad \text{and} \quad (\mathrm{Nul}(A))^\perp = \mathrm{Col}(A^T)$$

Proof $A = (\underline{a}_1 \dots \underline{a}_n) \Rightarrow A^T = \begin{pmatrix} \underline{a}_1^T \\ \vdots \\ \underline{a}_n^T \end{pmatrix}$

$\underline{u}$ in $(\mathrm{Col}(A))^\perp \Leftrightarrow \underline{a}_i \cdot \underline{u} = 0$ for all i

$\Leftrightarrow \underline{a}_i^T \underline{u} = 0$ for all i

$$\iff \quad A^T \underline{u} = \underline{0} \quad \iff \quad \underline{u} \text{ in } \mathrm{Nul}(A^T)$$

$$\Rightarrow \quad (\mathrm{Col}(A))^\perp = \mathrm{Nul}(A^T)$$

Recall $(A^T)^T = A \quad \Rightarrow \quad (\mathrm{Col}(A^T))^\perp = \mathrm{Nul}(A)$

$$= \quad (\mathrm{Nul}(A))^\perp = \mathrm{Col}(A^T)$$

$\square$