Inner Product Spaces

Aim: Extend theory of standard inner product to more general vector spaces.

Recall

Standard Inner Product on $\mathbb{R}^n$ = Function that, to each pair of vectors $\underline{u}, \underline{v}$ in $\mathbb{R}^n$, assigns a number $\underline{u} \cdot \underline{v}$

Properties we've used repeatedly:

1/ $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$
2/ $(\underline{u} + \underline{v}) \cdot \underline{w} = \underline{u} \cdot \underline{w} + \underline{v} \cdot \underline{w}$
3/ $(\lambda \underline{u}) \cdot \underline{v} = \lambda (\underline{u} \cdot \underline{v})$
4/ $\underline{u} \cdot \underline{u} \geqslant 0$ and $\underline{u} \cdot \underline{u} = 0 \Leftrightarrow \underline{u} = \underline{0}$

Everything we've done comes from these properties and fact $\mathbb{R}^n$ is finite dimensional

Definition Let V be a vector space. An inner product on V is a function that, to each pair of vector $\underline{u}, \underline{v}$ in V, assigns a real number $\langle \underline{u}, \underline{v} \rangle$ such that

1/ $\langle \underline{u}, \underline{v} \rangle = \langle \underline{v}, \underline{u} \rangle$
2/ $\langle (\underline{u} + \underline{v}), \underline{w} \rangle = \langle \underline{u}, \underline{w} \rangle + \langle \underline{v}, \underline{w} \rangle$
3/ $\langle (\lambda \underline{u}), \underline{v} \rangle = \lambda \langle \underline{u}, \underline{v} \rangle$
4/ $\langle \underline{u}, \underline{u} \rangle \geqslant 0$ and $\langle \underline{u}, \underline{u} \rangle = 0 \Leftrightarrow \underline{u} = \underline{0}$

Inner Product Space = Vector space together with inner product

Fundamental Example: $\mathbb{R}^n$ together with standard inner product.

Remark We can extend all familiar terminology from $\mathbb{R}^n$ to any inner product space:

$\|\underline{v}\| := \sqrt{\langle \underline{v}, \underline{v} \rangle}$ = length (or norm) of $\underline{v}$

$\|\underline{u} - \underline{v}\|$:= distance between $\underline{u}$ and $\underline{v}$

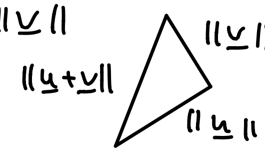
$\underline{u}, \underline{v}$ orthogonal in V $\iff$ $\langle \underline{u}, \underline{v} \rangle = 0$

<u>Remarkable Fact</u> Everything we've developed for $\mathbb{R}^n$ (e.g. orthogonal complements, projections, Gram-Schmidt) also holds for V a finite dimensional inner product space.

Important General Properties :

1) Cauchy - Schwarz Inequality $|\langle \underline{u}, \underline{v} \rangle| \leq \|\underline{u}\| \|\underline{v}\|$

2) Triangle Inequality : $\|\underline{u} + \underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$



<u>Examples</u>

1) $V = \mathbb{P}_n(\mathbb{R})$ and fix $t_0, t_1, \ldots, t_n$ distinct real numbers

Define

$$\langle p, q \rangle := p(t_0)q(t_0) + p(t_1)q(t_1) + \cdots + p(t_n)q(t_n)$$

Fact : $\langle \, , \rangle$ gives an inner product on $\mathbb{P}_n(\mathbb{R})$

4) : $\langle p, p \rangle = (p(t_0))^2 + \cdots + (p(t_n))^2 \geq 0$

$\langle p, p \rangle = (p(t_0))^2 + \cdots + (p(t_n))^2 = 0$ n+1 distinct zeroes

$\iff$ $p(t_0) = p(t_1) = \cdots = p(t_n) = 0$

<u>Fact:</u> Non-zero degree n polynomial can have at most n distinct zeroes

$\Rightarrow$ p = zero polynomial.

2/ $\langle \, , \, \rangle$ = inner product on $\mathbb{R}^n$ ← perhaps non-standard

1/, 2/, 3/ $\Rightarrow$ If we know $\langle \underline{e}_i, \underline{e}_j \rangle$ for all i and j we know $\langle \underline{u}, \underline{v} \rangle$ for all $\underline{u}, \underline{v}$ in $\mathbb{R}^n$

Example $\left\langle \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle = \langle 2\underline{e}_1 + 3\underline{e}_2, \underline{e}_1 + \underline{e}_2 \rangle$

$$= 2\langle \underline{e}_1, \underline{e}_1 \rangle + 2\langle \underline{e}_1, \underline{e}_2 \rangle + 3\langle \underline{e}_2, \underline{e}_1 \rangle + 3\langle \underline{e}_2, \underline{e}_2 \rangle$$

Define $A = \begin{pmatrix} \langle \underline{e}_1, \underline{e}_1 \rangle & \cdots & \langle \underline{e}_1, \underline{e}_n \rangle \\ \vdots & & \\ \langle \underline{e}_n, \underline{e}_1 \rangle & \cdots & \langle \underline{e}_n, \underline{e}_n \rangle \end{pmatrix}$

Properties : $\langle \underline{e}_i, \underline{e}_j \rangle = \langle \underline{e}_j, \underline{e}_i \rangle \Rightarrow A = A^T$

<u>Awesome Fact</u> : $\langle \underline{u}, \underline{v} \rangle = \underline{u}^T A \, \underline{v}$

$\langle \, , \, \rangle$ = standard inner product $\Leftrightarrow A = I_n$

<u>Even More Awesome Fact</u> :

Given A – $n \times n$ matrix

$\langle \underline{u}, \underline{v} \rangle = \underline{u}^T A \underline{v}$ an inner product $\Longleftrightarrow$ $A = A^T$ **and** eigenvalues of A are strictly positive ← We'll see why later

3/ $V = C[a,b]$ = continuous functions on $[a,b]$.

$$\langle f(x), g(x) \rangle := \int_a^b f(x) g(x) \, dx$$ ← Need Calculus to show it's an inner product

If $W = \text{Span}(1, x^2) \subset C[-1,1]$

Find $\text{Proj}_W(x^3)$.

Must first apply G.S. to $\{1, x^2\}$

$$\underline{v}_1 = 1$$

$$\underline{v}_2 = x^2 - \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle} 1 \qquad \left(\int_{-1}^{1} x^2 dx = \frac{2}{3} \ , \ \int_{-1}^{1} 1 \cdot 1 \ dx = 2 \right)$$

$$= x^2 - \frac{1}{3}$$

$\Rightarrow \{1, x^2 - \frac{1}{3}\}$ = orthogonal basis for $\text{Span}(1, x^2)$

$$\Rightarrow \text{Proj}_W(x^3) = \frac{\langle x^3, 1\rangle}{\langle 1, 1\rangle} 1 + \frac{\langle x^3, x^2 - \frac{1}{3}\rangle}{\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3}\rangle} x^2 - \frac{1}{3}$$

$$\int_{-1}^{1} x^3 \cdot 1 dx = 0, \qquad \int_{-1}^{1} x^3 (x^2 - \tfrac{1}{3}) \, dx = \frac{2}{7}$$

$$\int_{-1}^{1} (x^2 - \tfrac{1}{3})(x^2 - \tfrac{1}{3}) \, dx = \frac{1}{5} x^5 - \frac{2}{9} x^3 + \frac{1}{9} x \Big|_{-1}^{1}$$

$$= \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{8}{45}$$

$$\Rightarrow \text{Proj}_W(x^3) = \frac{\left(\frac{2}{7}\right)}{\left(\frac{8}{45}\right)} (x^2 - \tfrac{1}{3}) = \frac{90}{56} (x^2 - \tfrac{1}{3})$$