Fourier Series

Theory of Taylor Series = Approximating a function near $x = a$ by polynomials.

For n large and x "near" a

$$f(x) \simeq f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Under favorable circumstances

convergent infinite series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

Theory of Fourier Series = Approximating a function on closed interval by sine and cosine functions of increasing frequency.

finitely many jump discontinuities

Convention : f is piecewise-continuous function on $[-L, L]$

A function g is $2L$-periodic $\Leftrightarrow$ $g(x) = g(x+2L)$ for all x

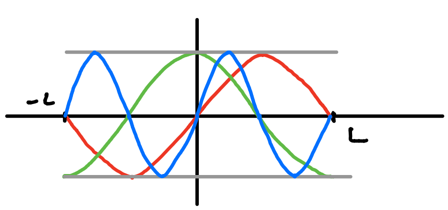
Important Examples :

$m = 0, 1, 2, \dots$

$\sin\left(\frac{m\pi x}{L}\right)$ and $\cos\left(\frac{m\pi x}{L}\right)$ are $2L$-periodic.

cosine is 2π-periodic

$$\left(\cos\left(\frac{m\pi(x+2L)}{L}\right) = \cos\left(\frac{m\pi x}{L} + 2\pi m\right) = \cos\left(\frac{m\pi x}{L}\right) \right)$$



— $= \sin\left(\frac{\pi x}{L}\right)$

— $= \cos\left(\frac{\pi x}{L}\right)$

— $= \sin\left(\frac{2\pi x}{L}\right)$

m = number of complete osscilations over $[-L, L]$ (the frequency)

Recall : If

guarantees integrable

$$V = \{ f : [-L, L] \longrightarrow \mathbb{R} \text{ , piecewise continuous} \}$$

then V has the inner-product

$$\langle f, g \rangle := \int_{-L}^{L} f(x)\, g(x)\, dx$$

constant function with value 1

Theorem $\left\{ \cos\left(\frac{0\pi x}{L}\right), \sin\left(\frac{\pi x}{L}\right), \cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \ldots \right\}$

is an orthogonal set.

Proof (outline)

$\sin\left(\frac{m\pi x}{L}\right)$ = odd function , $\cos\left(\frac{n\pi x}{L}\right)$ = even function

$\Rightarrow$ $\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)$ = odd function $\Rightarrow$

$$\left\langle \sin\left(\frac{m\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right) \right\rangle = \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) dx = 0$$

Integration by Parts (Twice) $\Rightarrow$

$$\left\langle \sin\left(\frac{m\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right) \right\rangle = \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases}$$

$$\left\langle \cos\left(\frac{m\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right) \right\rangle = \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \neq 0 \\ 2L & \text{if } m = n = 0 \end{cases}$$

$\square$

Imagine now $f(x) = \frac{a_0}{2} + \sum_{m=1}^{k} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right)$

(Example : $f(x) = \frac{5}{2} + \left(6\cos\left(\frac{\pi x}{L}\right) - 1\sin\left(\frac{\pi x}{L}\right)\right) + \left(0\cos\left(\frac{2\pi x}{L}\right) + 2\sin\left(\frac{2\pi x}{L}\right)\right)$)

Q: Can we determine a_m, b_m in a direct way ?

Observation :
$(n \geq 0)$

$$\left\langle f(x), \cos\left(\frac{n\pi x}{L}\right) \right\rangle$$

$$= \left\langle \frac{a_0}{2} + \sum_{m=1}^{k} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right), \cos\left(\frac{n\pi x}{L}\right) \right\rangle$$

$$= \frac{a_0}{2} \left\langle \cos\left(\frac{0\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right) \right\rangle + \sum_{m=1}^{k} a_m \left\langle \cos\left(\frac{m\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right) \right\rangle + \sum_{m=1}^{k} b_m \left\langle \sin\left(\frac{m\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right) \right\rangle$$

$$= \begin{cases} \frac{a_0}{2} \cdot 2L & \text{if } n = 0 \\ a_n \cdot L & \text{if } 0 < n \leq k \\ 0 & \text{if } n > k \end{cases}$$

Similarly, $\left\langle f(x), \sin\left(\frac{n\pi x}{L}\right) \right\rangle = \begin{cases} b_n \cdot L & \text{if } 0 < n \leq k \\ 0 & \text{if } n > k \end{cases}$

Conclusion :

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{k} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right)$$

$$\Rightarrow \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \quad \text{for } 0 \leq n \leq k$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } 0 < n \leq k$$

Remark : Similar to fact that $f(x) = c_0 + c_1 x + \cdots + c_k x^k$

$$\Rightarrow \quad c_n = \frac{f^{(n)}(0)}{n!}$$

Definition Let f be piecewise-continuous on $[-L, L]$, then the Fourier series of f is the infinite series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

where

$$a_n := \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx \quad , \quad n = 0, 1, 2, \dots$$

$$b_n := \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad , \quad n = 1, 2, \dots$$

Remark We've proven that if $f(x)$ is a trigonometric sum then the Fourier series is a finite sum and is equal to $f(x)$ for all x in $[-L, L]$.

$L = 1$

Example Compute Fourier Series of $f(x) = |x|$ on $[-1, 1]$

f even $\Rightarrow$ $f(x) \sin(n\pi x)$ odd $\Rightarrow$ $\displaystyle\int_{-1}^{1} f(x) \sin(n\pi x) \, dx = 0 = b_n$ for all $n \geq 1$

$f(x) \cos(n\pi x)$ even $\Rightarrow$ $\displaystyle\int_{-1}^{1} f(x) \cos(n\pi x) \, dx = 2 \int_{0}^{1} f(x) \cos(n\pi x) \, dx$

$$\Rightarrow \quad a_n = 2 \int_0^1 x \cos(n\pi x) \, dx \quad \text{for} \quad n \geq 0$$

$$\Rightarrow \quad a_0 = 2 \int_0^1 x \, dx = x^2 \Big|_0^1 = 1$$

Integration by parts: $(n \geq 1)$

$$\int_0^1 x \cos(n\pi x) \, dx = \cancelto{0}{\frac{1}{n\pi} x \sin(n\pi x) \Big|_0^1} - \int_0^1 \frac{1}{n\pi} \sin(n\pi x) \, dx$$

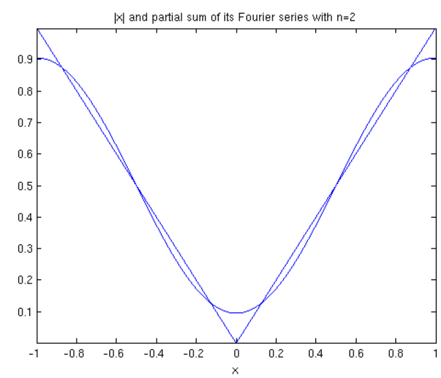
$$= \frac{1}{(n\pi)^2} \cos(n\pi x) \Big|_0^1 = \frac{1}{(n\pi)^2} \left((-1)^n - 1 \right)$$

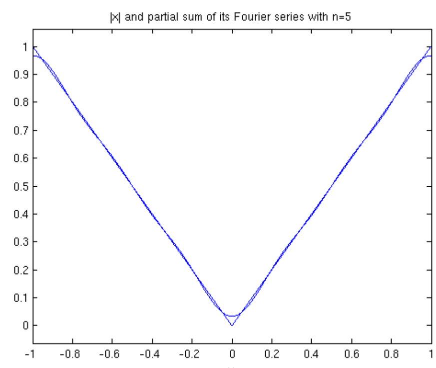
$$\Rightarrow \quad a_n = \begin{cases} 1 & \text{if } n = 0 \\ \frac{2}{(n\pi)^2}\left((-1)^n - 1\right) & \text{if } n > 0 \end{cases}$$

$\Rightarrow$ Fourier Series of $|x|$ on $[-1, 1]$ is

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2} \left((-1)^n - 1\right) \cos(n\pi x)$$

What happens when we look at some partial sums?





The more terms in the partial sum the closer to $|x|$ the series gets.

<u>Theorem</u> If f and f' are piecewise-continuous on $[-L, L]$ then for any x in $(-L, L)$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right) = \frac{1}{2}\left(f(x^+) + f(x^-) \right)$$

where $f(x^+) = \lim_{h \to 0^+} f(x+h)$, $f(x^-) = \lim_{h \to 0^-} f(x+h)$

For $x = \pm L$ the series converges to $\frac{1}{2}\left(f(-L^+) + f(L^-) \right)$

Consequences:

1/ If f continuous on $[-L,L]$ and $f(-L) = f(L)$ then Fourier series converges to $f(x)$ for all x in $[-L,L]$.

2/ If f is a continuous $2L$-periodic function on $(-\infty,\infty)$ then Fourier series converges to $f(x)$ for all x in $(-\infty,\infty)$.

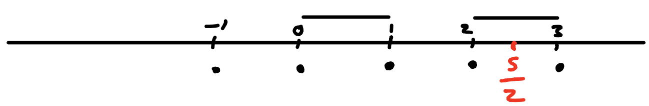
Example $f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ -2 & \text{if } -1 \leq x < 0 \end{cases}$ on $[-1,1]$. What does the Fourier series converge to at 0? What about at $\frac{5}{2}$?

First note that f' is piecewise continuous on $[-1,1]$.

$\Rightarrow$ Fourier series converges to $\frac{1}{2}(f(0^+) + f(0^-))$ at $x=0$

$f(0^+) = 1$, $f(0^-) = -2$ $\Rightarrow$ Fourier series converges to $\frac{-1}{2}$ at $x=0$.

Fourier series is 2-periodic $\Rightarrow$ Fourier series converges to 1 at $\frac{5}{2}$.



Important Fact: We can differentiate and integrate a Fourier series term by term.

assumes $F'(x)$, $F''(x)$ cts

ie $$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

$$\Rightarrow F'(x) = \sum_{n=1}^{\infty}\left(-\frac{n\pi}{L} a_n \sin\left(\frac{n\pi x}{L}\right) + \frac{n\pi}{L} b_n \cos\left(\frac{n\pi x}{L}\right)\right)$$

$$\int F(x)\,dx = C + \frac{a_0 x}{2} + \sum_{n=1}^{\infty}\left(\frac{L}{n\pi} a_n \sin\left(\frac{n\pi x}{L}\right) - \frac{L}{n\pi} b_n \cos\left(\frac{n\pi x}{L}\right)\right)$$

Fourier Sine and Cosine Series

Only cosine terms in Fourier series

Observation : f even on $[-L, L] \Rightarrow b_n = 0$ for all n

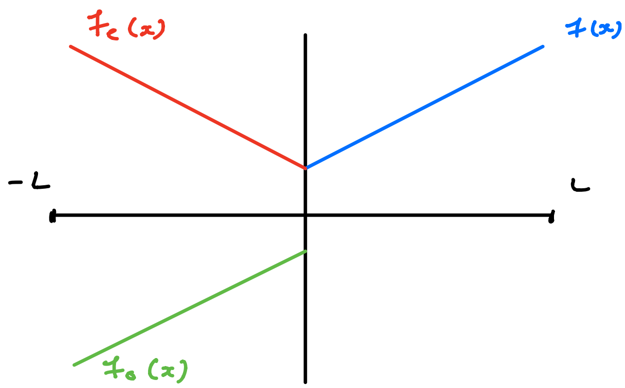
f odd on $[-L, L] \Rightarrow a_n = 0$ for all n

Only sine terms in Fourier series.

f - piecewise continuous function on $[0, L]$

Define
$$f_e(x) = \begin{cases} f(x) & \text{if } 0 \le x \le L \\ f(-x) & \text{if } -L \le x \le 0 \end{cases}$$

$$f_o(x) = \begin{cases} f(x) & \text{if } 0 < x \le L \\ -f(-x) & \text{if } -L < x < 0 \end{cases}$$



Definition

The Fourier cosine series for f on $[0, L]$ is the Fourier series of f_e on $[-L, L]$.

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where
$$a_n = \frac{1}{L}\int_{-L}^{L} \overbrace{f_e(x) \cos\left(\frac{n\pi x}{L}\right)}^{\text{even}} dx = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right).$$

The Fourier sine series of f on $[0, L]$ is the Fourier series of f_o on $[-L, L]$.

$$\sum_{n=1}^{\infty} b_k \sin\left(\frac{n\pi x}{L}\right)$$

where $$b_k = \frac{1}{L}\int_{-L}^{L} \overbrace{f_o(x)\sin\left(\frac{n\pi x}{L}\right)}^{\text{even}} dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$