Final Exam Review 3

$ay'' + by' + cy = f(t)$ — 2^{nd}-order, linear constant-coefficient differential equation.

$f(t)$ = zero function $\Rightarrow$ Homogeneous

Fact: Solution to $ay''+by'+cy=0$ are a two dimensional vector space

$ay''+by'+cy=0 \rightsquigarrow ax^2+bx+c=0 \leftarrow$ auxiliary equation

r a solution of auxiliary equation $\Rightarrow y(t) = e^{rt}$ a homogeneous solution.

3 cases

1) $r_1 \neq r_2$ are non-repeated roots of auxiliary equation $\Rightarrow c_1 e^{r_1 t} + c_2 e^{r_2 t}$ general homogeneous solution

2) r_1 a repeated root of auxiliary equation $\Rightarrow c_1 e^{r_1 t} + c_2 t e^{r_1 t}$ general homogeneous solution

3) $\alpha \pm i\beta$ non-real roots of auxiliary equation $\Rightarrow c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)$ general homogeneous solution

Fact: If y_p is a particular solution to $ay''+by'+cy = \underset{\text{perhaps non-zero}}{f(t)}$

$\Rightarrow y_p + y_h$ is a general solution

$y_h \leftarrow$ general homogeneous solution.

One method to find y_p : Undetermined coefficients

$$ay''+by'+cy = \begin{cases} \underset{\text{degree } m \text{ polynomial}}{P_m(t)} e^{kt} \\ P_m(t) e^{kt}\cos(\ell t) \\ P_m(t) e^{kt}\sin(\ell t) \end{cases}$$

For $P_m(t)e^{kt}$ → Try $y_p(t) = t^s(A_0 + \dots + A_m t^m)e^{kt}$

$ak^2+bk+c \neq 0 \Rightarrow s=0$

$ak^2+bk+c=0$ (non-repeated) $\Rightarrow s=1$

$ak^2+bk+c=0$ (repeated) $\Rightarrow s=2$

For $P_m(t)e^{kt}\cos(\ell t)$, $P_m(t)e^{kt}\sin(\ell t)$ → Try

$$y_p(t) = t^s(A_0+\dots+A_m t^m)e^{kt}\cos(\ell t) + t^s(B_0+\dots+B_m t^m)e^{kt}\sin(\ell t)$$

$a(k+i\ell)^2 + b(k+i\ell)+c \neq 0 \Rightarrow s=0$

$a(k+i\ell)^2 + b(k+i\ell)+c = 0 \Rightarrow s=1$

<u>Fact</u> : If $f(t)$ is a sum of such functions we can find a particular solution for each and add them (superposition principle)

Linear Systems of First-Order Differential Equations

$$\underline{x} : \mathbb{R} \longrightarrow \mathbb{R}^n \qquad \leftarrow \mathbb{R}^n\text{-valued function}$$
$$t \longmapsto \underline{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$\left(\underline{x} = \underline{y} \iff \underline{x}(t) = \underline{y}(t) \text{ for all } t \right)$

<u>Fact</u>: $\mathbb{R}^n$-valued functions form a vector space. $\left(\underline{0} = \text{zero function, ie } \underline{0}(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \text{ for all } t \right)$

$$\underline{x}'(t) := \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} \quad , \quad A - n \times n \text{ matrix}$$

$\underline{x}'(t) = A\underline{x}(t)$ — $n \times n$, homogeneous, constant-coefficient, linear system of first-order differential equations.

<u>Facts</u> :

1/ $\underline{x}, \underline{y}$ solutions and $\underline{x}(t_0) = \underline{y}(t_0)$ for some $t_0 \Rightarrow \underline{x} = \underline{y}$

2/ Solutions form an n-dimensional vector space.

General Facts about any $\mathbb{R}^n$-valued functions :

$\{\underline{x}_1, \dots, \underline{x}_n\}$ L.D. $\Rightarrow$ $\det(\underline{x}_1(t) \dots \underline{x}_n(t)) = 0$ <u>for all t</u>

$\not\Leftarrow$ ← not in general

Hence,

If there exists t_0 such that $\det(\underline{x}_1(t_0) \dots \underline{x}_n(t_0)) \neq 0$ $\rightarrow$ $\{\underline{x}_1, \dots, \underline{x}_n\}$ L.I.

Problem :

$\det(\underline{x}_1(t) \dots \underline{x}_n(t)) = 0$ <u>for all t</u> tells us nothing.

Partial Converse : $\{\underline{x}_1, \dots, \underline{x}_n\}$ Solutions to $\underline{x}'(t) = A\underline{x}(t)$

L.I. $\Updownarrow$ $\det(\underline{x}_1(t) \dots \underline{x}_n(t)) \neq 0$ <u>for all t</u>

L.D. $\Updownarrow$ $\det(\underline{x}_1(t) \dots \underline{x}_n(t)) = 0$ <u>for all t</u>

Important Observation:

$\underline{v}$ an eigenvector of A with eigenvalue λ $\Rightarrow$ $\underline{x}(t) = e^{\lambda t}\underline{v}$ a solution.

vectors in $\mathbb{R}^n$

$\{\underline{v}_1, \ldots, \underline{v}_n\}$ an eigenbasis of A $\Rightarrow$ $\underline{x}(t) = c_1 e^{\lambda_1 t}\underline{v}_1 + \cdots + c_n e^{\lambda_n t}\underline{v}_n$

a general solution

If $\alpha + i\beta$ a non-real eigenvalue of A (a non-real root of characteristic polynomial)

We can still find complex eigenvectors by calculating

$$\mathrm{Null}\,(A - (\alpha + i\beta) I_n)$$

Do usual row reduction by doing complex arithmetic

in $\mathbb{R}^n$

$\underline{a} + i\underline{b}$ a complex eigenvector with complex eigenvalue $\alpha + i\beta$

$$\Rightarrow \quad e^{\alpha t}\cos(\beta t)\,\underline{a} - e^{\alpha t}\sin(\beta t)\,\underline{b}, \quad e^{\alpha t}\sin(\beta t)\,\underline{a} + e^{\alpha t}\cos(\beta t)\,\underline{b}$$

are L.I. solutions.

Fourier Series

f - piecewise continuous function on $[-L, L]$

waves at increasing frequency

$$\text{F.S. of } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

amplitudes

$$\text{where} \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1, 2, 3, \ldots$$

usually calculated using I.B.P.

f even $\Rightarrow$ $b_n = 0$ for all $n = 1, 2, 3, \ldots$

f odd $\Rightarrow$ $a_n = 0$ for all $n = 0, 1, 2, 3 \ldots$

Cosine Fourier Series of $f(x)$ on $[0, L] = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$

even extension to $[-L, L]$

where $a_n = \frac{1}{L}\int_{-L}^{L} f_e(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$

Sine Fourier Series of $f(x)$ on $[0, L] = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

odd extension to $[-L, L]$

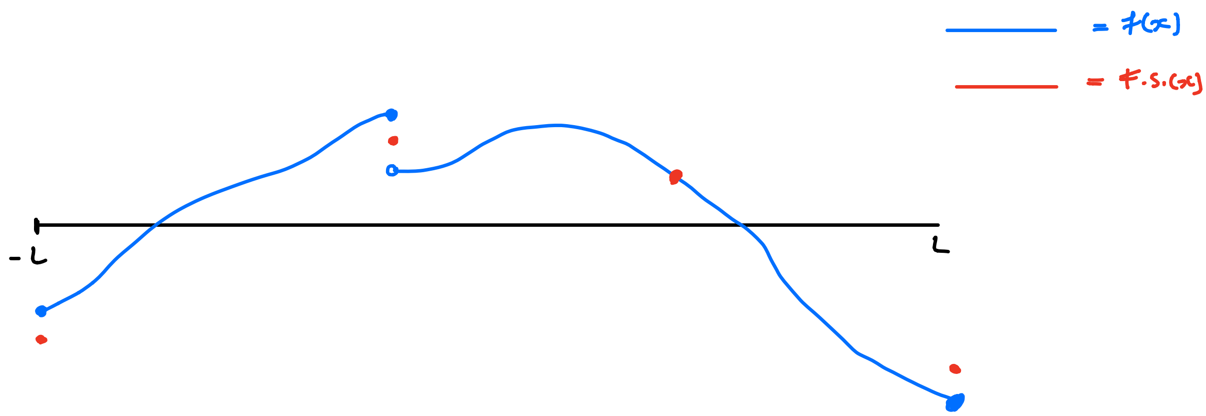
where $b_n = \frac{1}{L}\int_{-L}^{L} f_o(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

<u>Convergence of F.S.</u>

value from above

value from below

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right) = \begin{cases} \dfrac{f(x^+) + f(x^-)}{2} & -L < x < L \\[2ex] \dfrac{f(-L^+) + f(L^-)}{2} & x = \pm L \end{cases}$$



Finally F.S. is $2L$-periodic $\Rightarrow$ F.S.(x) = F.S.$(x+2L)$ for all x.