Final Review 2

A - $n \times n$ matrix

Called an eigenvector.

λ in $\mathbb{R}$ is an eigenvalue of A $\iff$ There exists $\underline{v}$ in $\mathbb{R}^n$, $\underline{v} \neq \underline{0}$ such that $A\underline{v} = \lambda \underline{v}$

λ - eigenspace $= \{\underline{v}$ in $\mathbb{R}^n$ such that $A\underline{v} = \lambda\underline{v}\} = Nul(A - \lambda I_n)$

variable

degree n polynomial

Characteristic polynomial of A $= \det(A - x I_n)$

λ is an eigenvalue of A $\iff$ $\dim(Nul(A - \lambda I_n)) > 0$

$\iff \det(A - \lambda I_n) = 0$

Algebraic multiplicity of λ = How many times λ is a root of $\det(A - x I_n)$

alg. mult. 2

alg. mult. 1

E.g. $\det(A - xI_3) = -(x-1)^2(x-2)^1 \Rightarrow 1, 2$ are eigenvalues

Important Definition :

Called eigenbasis

A <u>diagonalizable</u> $\iff$ There exists a basis $\{\underline{v}_1, ..., \underline{v}_n\}$ of $\mathbb{R}^n$ consisting of eigenvectors of A (ie $A\underline{v}_i = \lambda_i \underline{v}_i$ for all i)

Facts

A similar to diagonal matrix

1/ $\mathcal{B} = \{\underline{v}_1, ..., \underline{v}_n\}$ eigenbasis $\iff$ $A_{\mathcal{B},\mathcal{B}} = P^{-1}AP = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

where $P = (\underline{v}_1 \ldots \underline{v}_n)$

2/ A diagonalizable $\iff$ $\dim(Nul(A - \lambda I_n)) =$ Algebraic multiplicity of λ

for all eigenvalues λ

Strategy to Diagonalize A :

a) Find all eigenvalues
b) For each λ, an eigenvalue find basis for $Nul(A - \lambda I_n)$
c) Take union of each basis

Inner Products, Norms and Orthogonality

Inner product on vector space V = Function that assigns to any pair $\underline{u}, \underline{v}$ in V a number $\langle \underline{u}, \underline{v} \rangle$ satisfying following properties :

1/ $\langle \underline{u}, \underline{v} \rangle = \langle \underline{v}, \underline{u} \rangle$

2/ $\langle \underline{u} + \underline{v}, \underline{w} \rangle = \langle \underline{u}, \underline{w} \rangle + \langle \underline{v}, \underline{w} \rangle$

3/ $\langle \lambda \underline{u}, \underline{w} \rangle = \lambda \langle \underline{u}, \underline{w} \rangle$

4/ $\langle \underline{u}, \underline{u} \rangle \geq 0$ and $\langle \underline{u}, \underline{u} \rangle = 0 \iff \underline{u} = \underline{0}_V$

Important Examples

standard inner product on $\mathbb{R}^n$

$V = \mathbb{R}^n$, $\langle \underline{u}, \underline{v} \rangle = \underline{u} \cdot \underline{v} = u_1 v_1 + \dots + u_n v_n$

$V = \{ f : [a,b] \longrightarrow \mathbb{R}, \text{ integrable} \}$, $\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx$

norm/length of $\underline{u}$

$\| \underline{u} \| = \sqrt{\langle \underline{u}, \underline{u} \rangle}$

$\underline{u}, \underline{v}$ orthogonal $\iff \langle \underline{u}, \underline{v} \rangle = 0$

Orthogonal complement of W in V

$W \subset V$ subspace $\Rightarrow W^{\perp} = \{ \underline{v} \text{ in } V \text{ such that } \langle \underline{v}, \underline{w} \rangle = 0 \text{ for all } \underline{w} \text{ in } W \}$

$W^{\perp}$ subspace and $\dim(V) < \infty \Rightarrow \dim(W) + \dim(W^{\perp}) = \dim(V)$

Important Examples : $\mathrm{Col}(A)^{\perp} = \mathrm{Nul}(A^T)$, $\mathrm{Nul}(A)^{\perp} = \mathrm{Col}(A^T)$

$\{ \underline{v}_1, \dots, \underline{v}_k \}$ orthogonal $\iff \langle \underline{v}_i, \underline{v}_j \rangle = 0$ for all $i \neq j$

$\{ \underline{v}_1, \dots, \underline{v}_k \}$ orthonormal $\iff \langle \underline{v}_i, \underline{v}_j \rangle = 0$ for all $i \neq j$ and $\| \underline{v}_i \| = 1$ for all i

Facts

1/ $\{ \underline{v}_1, \dots, \underline{v}_k \}$ orthogonal and all non-zero $\Rightarrow \{ \underline{v}_1, \dots, \underline{v}_k \}$ L.I.

2/ In this case

$$\underline{x} = \lambda_1 \underline{v}_1 + \dots + \lambda_k \underline{v}_k \Rightarrow \lambda_i = \frac{\langle \underline{x}, \underline{v}_i \rangle}{\langle \underline{v}_i, \underline{v}_i \rangle}$$

3/ $n \times k$ — orthonormal w.r.t. standard inner product on $\mathbb{R}^n$

$U = (\underline{v}_1 \dots \underline{v}_k) \Rightarrow U^T U = I_k$

$n = k \Rightarrow U^T U = U U^T = I_n$ — Call U an orthogonal matrix

V = vector space with inner product.

$W = \mathrm{Span}(\underline{v}_1, \dots, \underline{v}_k) \subset V$

non-zero, orthogonal basis

$$\mathrm{Proj}_W(\underline{x}) = \frac{\langle \underline{x}, \underline{v}_1 \rangle}{\langle \underline{v}_1, \underline{v}_1 \rangle} \underline{v}_1 + \dots + \frac{\langle \underline{x}, \underline{v}_k \rangle}{\langle \underline{v}_k, \underline{v}_k \rangle} \underline{v}_k$$

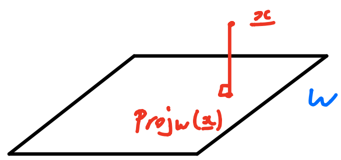
orthogonal projection of $\underline{x}$ onto W

Facts

1/ $\mathrm{Proj}_W(\underline{x})$ does not depend on choice of orthogonal basis

2/ $\underline{x} = \overbrace{\mathrm{Proj}_W(\underline{x})}^{\text{in } W} + \overbrace{\mathrm{Proj}_{W^\perp}(\underline{x})}^{W^\perp}$. This breakdown is unique

3/ $\mathrm{Proj}_W(\underline{x})$ is "closest" point in W to $\underline{x}$.



$\| \underline{x} - \mathrm{Proj}_W(\underline{x}) \| = \| \mathrm{Proj}_{W^\perp}(\underline{x}) \| =$ min distance between $\underline{x}$ and W

$$\{\underline{x}_1, \dots, \underline{x}_k\} \text{ a basis for } W \xrightarrow{\text{Gram-Schmidt}} \{\underline{v}_1, \dots, \underline{v}_n\} \text{ orthogonal basis for } W$$

$\underline{v}_1 = \underline{x}_1$

$$\underline{v}_2 = \underline{x}_2 - \frac{\langle \underline{x}_2, \underline{v}_1 \rangle}{\langle \underline{v}_1, \underline{v}_1 \rangle} \underline{v}_1$$

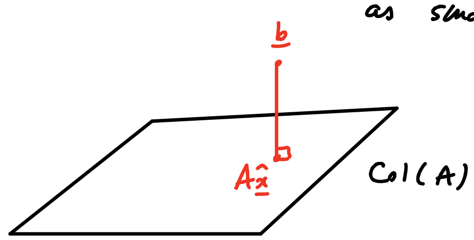
$\vdots$

$$\underline{v}_k = \underline{x}_k - \frac{\langle \underline{x}_k, \underline{v}_1 \rangle}{\langle \underline{v}_1, \underline{v}_1 \rangle} \underline{v}_1 - \frac{\langle \underline{x}_k, \underline{v}_2 \rangle}{\langle \underline{v}_2, \underline{v}_2 \rangle} \underline{v}_2 \dots - \frac{\langle \underline{x}_k, \underline{v}_k \rangle}{\langle \underline{v}_k, \underline{v}_k \rangle} \underline{v}_k$$

Fact : $\mathrm{Span}(\underline{x}_1, \dots, \underline{x}_i) = \mathrm{Span}(\underline{v}_1, \dots, \underline{v}_i)$ for all $i \leq k$

Least-Squares Problems : Given linear system $A\underline{x} = \underline{b}$, find $\hat{\underline{x}}$ such that $\|A\hat{\underline{x}} - \underline{b}\|$ is as small as possible.

standard inner product



$\hat{\underline{x}}$ is a least squares solution to $A\underline{x} = \underline{b}$

$$\Leftrightarrow \quad A\hat{\underline{x}} = \mathrm{Proj}_{\mathrm{Col}(A)}(\underline{b}) \quad \Leftrightarrow \quad A\hat{\underline{x}} - \underline{b} \text{ in } \mathrm{Col}(A)^{\perp}$$

$$\Leftrightarrow \quad A\hat{\underline{x}} - \underline{b} \text{ in } \mathrm{Nul}(A^T)$$

$$\Leftrightarrow \quad A^T A\hat{\underline{x}} = A^T\underline{b}$$

Called the normal equations of $A\underline{x} = \underline{b}$

Spectral Theorem and SVD.

A - $n \times n$ matrix symmetric $\Leftrightarrow$ $A = A^T$

A - $n \times n$ matrix orthogonally diagonalizable $\Leftrightarrow$ There exists P orthogonal such that

$$\overset{P^{-1}}{P^T} A P = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$\Leftrightarrow$ There exist an orthogonal basis of eigenvectors of A

Spectral Theorem : A symmetric $\Leftrightarrow$ A orthogonally diagonalizable

Overview of Finding SVD of A an $m \times n$ matrix

1. Orthogonally diagonalize $A^T A$ giving orthonormal basis $\{\underline{v}_1, \ldots, \underline{v}_n\}$
2. Reorder so that eigenvalues are decreasing $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$
3. Take square roots to give $\overbrace{\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r}^{\text{non-zero}} \geq \overbrace{\sigma_{r+1} \geq \cdots \geq \sigma_n}^{\text{zero}}$

4/ Define $\underline{u}_i = \frac{1}{\sigma_i} A \underline{v}_i$ for $1 \le i \le r$.

5/ Extend $\{\underline{u}_1, \dots, \underline{u}_r\}$ to orthonormal basis $\{\underline{u}_1, \dots, \underline{u}_r, \overbrace{\underline{u}_{r+1}, \dots, \underline{u}_m}^{\text{Basis for } \mathrm{Nul}(A^T)}\}$

6/ $U = (\underline{u}_1 \dots \underline{u}_m)$

$V = (\underline{v}_1 \dots \underline{v}_m)$

$$\Sigma = \left(\begin{array}{c|c} \begin{matrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right) \Rightarrow A = U \Sigma V^T$$

S.V.D.