# Final Review 1

Fundamental Concepts :

Vector Space = Set together with some addition and scalar multiplication satisfying 8 core axioms.

E.g. There is a zero vector $\underline{0}$ (or $\underline{0}_V$)

$(\underline{x}+\underline{y})+\underline{z} = \underline{x}+(\underline{y}+\underline{z})$

Main Examples : $\mathbb{R}^n$ , $C[a,b]$ - continuous $\mathbb{R}$-valued functions on $[a,b]$

$\mathbb{P}_n(\mathbb{R})$ - polynomials of degree at most $n$ with real coefficients.

vector spaces

Linear Transformation = Function $T : V \longrightarrow W$ such that

1) $T(\underline{x}+\underline{y}) = T(\underline{x}) + T(\underline{y})$

2) $T(\lambda \underline{x}) = \lambda T(\underline{x})$

in $\mathbb{R}^m$

$(\underline{a}_1 \dots \underline{a}_n)$

Main Example : $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, $A$ – $m \times n$ matrix

All $T : \mathbb{R}^n \to \mathbb{R}^m$ Linear are of this form

$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\underline{x} \longrightarrow A\underline{x} = x_1 \underline{a}_1 + x_2 \underline{a}_2 + \dots + x_n \underline{a}_n$

$A = (T(\underline{e}_1) \dots T(\underline{e}_n))$ , $\{\underline{e}_1, \dots, \underline{e}_n\} \subset \mathbb{R}^n$ standard basis

Linear Algebra = Study of Linear Transformations

Important Properties of subsets of a vector space $V$ :

1) $\{\underline{v}_1, \dots, \underline{v}_n\} \subset V$ L.I. if $c_1\underline{v}_1 + \dots + c_n\underline{v}_n = \underline{0} \Rightarrow c_1 = 0, \dots, c_n = 0$

$\{v_1, \dots, v_n\} \subset V$ L.D. if $c_1\underline{v}_1 + \dots + c_n\underline{v}_n = \underline{0} \not\Rightarrow c_1 = 0, \dots, c_n = 0$

ie. There exist $c_1, \ldots, c_n$, not all zero, such that $c_1\underline{v}_1 + \cdots + c_n\underline{v}_n = \underline{0}$

2) $\{\underline{v}_1, \ldots, \underline{v}_n\} \subset V$ spanning if $\text{Span}(\underline{v}_1, \ldots, \underline{v}_n) = \{\lambda_1 \underline{v}_1 + \cdots + \lambda_n \underline{v}_n\} = V$

3) $\{\underline{v}_1, \ldots, \underline{v}_n\} \subset V$ basis if <u>both</u> L.I. and spanning

<u>Key Facts</u>

a) Every L.I. subset can be extended to a basis

b) Every spanning set contains a basis

dimension of $V$

c) Any two bases have same size, denoted $\dim(V)$

e.g. $\{\underline{e}_1, \ldots, \underline{e}_n\} \subset \mathbb{R}^n$ standard basis $\Rightarrow \dim(\mathbb{R}^n) = n$

d) $\{\underline{v}_1, \ldots, \underline{v}_n\} \subset V$ a basis $\Rightarrow$ Given any $\underline{x}$ in $V$ there exist <u>unique</u> real numbers $\lambda_1, \ldots, \lambda_n$ such that $\underline{x} = \lambda_1 \underline{v}_1 + \cdots + \lambda_n \underline{v}_n$

$V = \{f : \mathbb{R} \to \mathbb{R}\}$, $f, g$ in $V$

$(f+g)(t) := f(t) + g(t)$

Need some function with following property:

equal as functions

$f + g = g$ for every $g$

$\Leftrightarrow$ $f(t) + g(t) = g(t)$ for all $t$ in $\mathbb{R}$ and all $g$ in $V$

Only option $f : \mathbb{R} \to \mathbb{R}$, $t \mapsto 0$ ← zero function

$0(t) = 0$ for all $t$ )

4/ $U \subset V$ a subspace if

a) $\underline{0}_V$ in $U$

b) $\underline{u}, \underline{v}$ in $U \Rightarrow \underline{u} + \underline{v}$ in $U$

c) $\underline{u}$ in $U$, $\lambda$ in $\mathbb{R} \Rightarrow \lambda \underline{u}$ in $U$

Facts

1/ $U \subset V$ subspace is itself a vector space

2/ $\mathrm{Span}(\underline{v}_1, \dots, \underline{v}_n) \subset V$ is always a subspace

3/ $U \subset V$ subspace $\Rightarrow \dim(U) \leq \dim(V)$

$\dim(U) = \dim(V) \Leftrightarrow U = V$

$\mathbb{Z}$ = integers = $\{\dots -2, -1, 0, 1, 2, 3, \dots\}$

$$\mathbb{Z}^n = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \ a_i \text{ in } \mathbb{Z} \right\}$$

Not a vector space because there is no real scalar multiplication

$$\frac{1}{2}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = ?$$

Special Case : Subsets of $\mathbb{R}^m$, $A = (\underline{a}_1 \dots \underline{a}_n)$ in $\mathbb{R}^m$

$\{\underline{a}_1, \dots, \underline{a}_n\}$ L.I. $\Leftrightarrow$ Reduced $A$ has pivot in every column $\Leftrightarrow T_A$ one-to-one

$\{\underline{a}_1, \dots, \underline{a}_n\}$ Spanning $\Leftrightarrow$ Reduced $A$ has pivot in every $\Leftrightarrow T_A$ onto

$\text{Rank}(T_A) = \text{Rank}(A)$ row

$$\text{Dim}\Big(\underbrace{\text{Span}(\underline{a}_1, \ldots, \underline{a}_n)}\Big) = \text{Number of pivot columns in reduced } A$$

Basis given by original pivot column of $A$

$\text{Range}(T_A) = \text{Col}(A)$

$$\text{Nul}(A) = \{\underline{x} \text{ in } \mathbb{R}^n \text{ such that } A\underline{x} = \underline{0}\} \subset \mathbb{R}^n$$

subspace

$$\text{Ker}(T_A) = \{\underline{x} \text{ in } \mathbb{R}^n \text{ such that } T_A(\underline{x}) = \underline{0}\}$$

$$\dim(\text{Nul}(A)) = \text{Number of Free columns of reduced } A$$

$$= \text{Nullity}(T_A)$$

Find a basis by writing general solution in parametric form with weights given by free variables.

$T_A : \mathbb{R}^n \to \mathbb{R}^m$

Rank-Nullity Theorem : $\dim(\text{Col}(A)) + \dim(\text{Nul}(A)) = n$

More generally : $T : V \to W$

$$\dim(\text{Range}(T)) + \dim(\text{Ker}(T)) = \dim(V)$$

Very Important : <u>Coordinates</u>

$\beta = \{\underline{b}_1, \ldots, \underline{b}_n\} \subset V$ a basis , $\underline{v}$ a vector in $V$

$$\underline{v} = \lambda_1 \underline{b}_1 + \lambda_2 \underline{b}_2 + \cdots + \lambda_n \underline{b}_n \iff (\underline{v})_\beta = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

$\underline{v}$ written in $\beta$-coordinates

Example $(\underline{v})_\beta = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \iff \underline{v} = 1 \cdot \underline{b}_1 + 2 \cdot \underline{b}_2 + 3 \cdot \underline{b}_3$

one-to-one, onto, Linear ← called an isomorphism

$V \longleftrightarrow \mathbb{R}^n$ Let's us identify $V$ with $\mathbb{R}^n$.

$\underline{v} \longleftrightarrow (\underline{v})_\beta$ Properties of subsets of $V$ translate to same properties in $\mathbb{R}^n$

Example $V = \mathbb{P}_2(\mathbb{R})$, $\beta = \{1, x, x^2\}$

$\mathbb{P}^2(\mathbb{R}) \longleftrightarrow \mathbb{R}^3$

$a_0 + a_1 x + a_2 x^2 \longleftrightarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$

abstract

$\beta = \{\underline{b}_1, \ldots, \underline{b}_n\}$, $C = \{\underline{c}_1, \ldots, \underline{c}_n\}$ bases for $V$.

$n \times n$ matrix

If $P_{C \leftarrow \beta} := \left( (\underline{b}_1)_C \cdots (\underline{b}_n)_C \right)$ then

$P_{C \leftarrow \beta} (\underline{v})_\beta = (\underline{v})_C$ ←

Special Case : $V = \mathbb{R}^n$

$\left( \underline{c}_1 \cdots \underline{c}_n \,\middle|\, \underline{v} \right) \longrightarrow \left( I_n \,\middle|\, (\underline{v})_C \right)$

$\left( \underline{c}_1 \cdots \underline{c}_n \,\middle|\, \underline{b}_1 \cdots \underline{b}_n \right) \longrightarrow \left( I_n \,\middle|\, P_{C \leftarrow \beta} \right)$

$(P_{C \leftarrow \beta})^{-1} = P_{\beta \leftarrow C}$

Coordinates and Linear Transformations

Fix $\mathcal{B} = \{\underline{b}_1, \ldots, \underline{b}_n\} \subset V$, $\quad \mathcal{C} = \{\underline{c}_1, \ldots, \underline{c}_m\} \subset W$ bases

$T : V \to W$ linear

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, leftrightarrow, "\mathcal{B}\text{-coordinates}"'] & W \arrow[d, leftrightarrow, "\mathcal{C}\text{-coordinates}"] \\
\mathbb{R}^n \arrow[r, "A_{\mathcal{B},\mathcal{C}}"] & \mathbb{R}^m
\end{tikzcd}$$

matrix associated to $T$ with respect to bases $\mathcal{B}$ and $\mathcal{C}$

$$\Rightarrow \quad A_{\mathcal{B},\mathcal{C}} (\underline{v})_{\mathcal{B}} = (T(\underline{v}))_{\mathcal{C}}$$

allows us to analyse $T$ concretely

<u>Facts</u>

1/ $A_{\mathcal{B},\mathcal{C}} = \left( (T(\underline{b}_1))_{\mathcal{C}} \ \cdots \ (T(\underline{b}_n))_{\mathcal{C}} \right)$ $\quad$ $m \times n$ matrix

E.g. $A_{\mathcal{B},\mathcal{C}} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 2 \end{pmatrix} \Rightarrow (T(\underline{b}_1))_{\mathcal{C}} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$\Rightarrow T(\underline{b}_1) = 1 \cdot \underline{c}_1 + 2 \cdot \underline{c}_2 + 1 \cdot \underline{c}_3$

$(T(\underline{b}_2))_{\mathcal{C}} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$

$\Rightarrow T(\underline{b}_2) = 2\underline{c}_1 + 3\underline{c}_2 + 2\underline{c}_3$

Major Misconception : $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}_{\mathcal{C}}$ $\longleftarrow$ not meaningful
not first column

2/ (Special Case) $\quad T = T_A : \mathbb{R}^n \to \mathbb{R}^m$
$\underline{v} \longrightarrow A\underline{v}$

$n \times n$

$$\Rightarrow \quad A_{\mathcal{B},\mathcal{C}} = Q^{-1} A P, \quad P = (\underline{b}_1 \dots \underline{b}_n), \quad Q = (\underline{c}_1 \dots \underline{c}_m) \leftarrow m \times m$$

<u>Example</u>

$$A = \begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}, \quad \mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$\Rightarrow \quad A_{\mathcal{B},\mathcal{C}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$$

<u>Fact</u>

$$\text{Rank}(A) = \text{Rank}(A_{\mathcal{B},\mathcal{C}})$$