DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

Formulae

$$\int \tan(x)\ dx = \ln|\sec(x)| + C \qquad \int \sec(x)\ dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \frac{1}{1+x^2} dx = \arctan(x) + C \qquad \int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + C$$

$$\frac{d\tan(x)}{dx} = \sec^2(x) \qquad \frac{d\sec(x)}{dx} = \tan(x)\sec(x)$$

$$1 = \sin^2(x) + \cos^2(x) \qquad 1 + \tan^2(x) = \sec^2(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$e^x = 1 + x + \tfrac{x^2}{2} + \tfrac{x^3}{6} + \cdots = \textstyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \tfrac{x^7}{7!} + \tfrac{x^9}{9!} - \cdots = \textstyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \tfrac{x^6}{6!} + \tfrac{x^8}{8!} - \cdots = \textstyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\arctan x = x - \tfrac{x^3}{3} + \tfrac{x^5}{5} - \tfrac{x^7}{7} + \tfrac{x^9}{9} - \cdots = \textstyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e$$

CALCULATORS ARE NOT PERMITTED

This exam consists of 10 questions. Answer the questions in the spaces provided.

Name and section: ______________________________

GSI's name: ______________________________

1. Compute the following integral:

   (a) (3 points)

$$\int x^3\sqrt{1-x^2}\, dx$$

   Solution:

(b) (7 points)

$$\int \cos(\ln(x))\ dx$$

Hint: try a substitution first.

Solution:

2. (10 points) Determine if the following series are absolutely convergent, conditionally convergent or divergent. You do not need to show your working.

   (a)
   $$\sum_{n=1}^{\infty} \frac{\cos(n) + \sin(n) + 3}{n^2}$$

   Solution:

   (b)
   $$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n} + 2}$$

   Solution:

   (c)
   $$\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$$

   Solution:

   (d)
   $$\sum_{n=1}^{\infty} \frac{10^n + 6^n}{3^n + 11^n + 5^n}$$

   Solution:

   (e)
   $$\sum_{n=1}^{\infty} (\frac{n}{n+1})^{n^2}$$

   Solution:

3. Determine the radii of convergence of the following power series:

(a) (5 points)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}(x+1)^n$$

Solution:

(b) (5 points)

$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(n!)^2} x^n$$

Solution:

4. (10 points) What is the domain of the function $f(x)$ given by the power series

$$\sum_{n=1}^{\infty} \frac{2^n (x+1)^n}{n^{1/3}}$$

What is the value of $f^{(8)}(-1)$?

Solution:

5. (10 points) Determine the Taylor series of the function

$$f(x) = (2 + x^2)^{1/3}$$

about the point $x = 0$. If you use the binomial theorem be sure you carefully define the binomial coefficients.

Solution:

6. (10 points) Find a solution to the initial-value problem

$$\cos(x)y' = y\sin(x), \quad y(0) = 1$$

Solution:

7. (10 points) Find the general solution to the following differential equation

$$xy' - y - \frac{x^2}{\sqrt{1+x^2}} = 0$$

Solution:

8. (10 points) Find the general equation of a curve which is orthogonal to the family of curves given by $y + kxy = x$, for k a constant.

Solution:

9. (10 points) Find the general solution to the following differential equation

$$y'' - 3y' + 2y = x(e^x + 1)$$

Solution:

10. (10 points) Find a non-zero power series solution to the following differential equation

$$y'' + xy' = 0.$$

You do not need to show convergence.

Solution:

END OF EXAM