Eigenvectors and Eigenvalues

A – $n \times n$ matrix.

Aim : Understand $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ in more depth.
$$\underline{x} \longmapsto A\underline{x}$$

Example $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

$$A\underline{e}_1 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\underline{e}_1$$

$$A\underline{e}_2 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3\underline{e}_2$$

$$A\underbrace{\underline{x}}_{= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = A(x_1\underline{e}_1 + x_2\underline{e}_2) = x_1 A\underline{e}_1 + x_2 A\underline{e}_2 = \underbrace{2x_1\underline{e}_1 + 3x_2\underline{e}_2}_{= \begin{pmatrix} 2x_1 \\ 3x_2 \end{pmatrix}}$$

Definition A – $n \times n$ matrix. An <u>eigenvector</u> of A is a vector $\underline{v}$ in $\mathbb{R}^n$ such that

1) $\underline{v} \neq \underline{0}$

2) $A\underline{v} = \lambda\underline{v}$ for some λ in $\mathbb{R}$

We call any such λ an <u>eigenvalue</u> of A.

In above example $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $\underline{e}_1$ is eigenvector with eigenvalue 2, $\underline{e}_2$ is eigenvector with eigenvalue 3.

Definition D – $n \times n$ matrix is <u>diagonal</u> if $(D)_{ij} = 0$ for all $i \neq j$. ie.

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

zero entries off leading diagonal.

Fact : A – $n \times n$ matrix diagonal $\iff$ $\{\underline{e}_1, \underline{e}_2, \dots, \underline{e}_n\}$ are all eigenvectors of A

$$A\underline{e}_i = \lambda_i \underline{e}_i \implies A = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

Q: How many eigenvectors correspond to a given eigenvalue?

A – $n \times n$ matrix, λ in $\mathbb{R}$

Definition The λ-eigenspace of A is the subset of $\mathbb{R}^n$ consisting of all eigenvectors with eigenvalue λ and the zerovector

Observation : $\underline{v}$ in λ–eigenspace $\iff A\underline{v} = \lambda \underline{v}$

$$\iff A\underline{v} - \lambda\underline{v} = \underline{0}$$
$$\iff (A - \lambda I_n)\underline{v} = 0$$
$$\iff \underline{v} \text{ in } \mathrm{Nul}(A - \lambda I_n)$$

Conclusion : λ-eigenspace of A $=$ $\mathrm{Nul}(A - \lambda I_n)$

$\implies$ λ-eigenspace of A is a subspace of $\mathbb{R}^n$

Observe : λ-eigenvalue of A $\iff$ λ-eigenspace $\neq \{\underline{0}\}$

Example

$$2\text{-eigenspace of } \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \mathrm{Nul}\left(\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right)$$
$$= \mathrm{Nul}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \implies x_3 = 0$$

$$\implies 2\text{-eigenspace of } \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \left\{\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}\right\} = \mathrm{Span}(\underline{e}_1, \underline{e}_2)$$

More generally :

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \Rightarrow \lambda\text{-eigenspace} = \text{Span of } \underline{e}_i \text{ such that } \lambda = \lambda_i .$$

Q: How do we find eigenvectors and eigenvalues if A not diagonal?

λ an eigenvalue of A $\iff$ There exists $\underline{v} \neq \underline{0}$ in $\mathbb{R}^n$ such that $A\underline{v} = \lambda \underline{v}$

$\iff$ There exists $\underline{v} \neq \underline{0}$ in $\mathbb{R}^n$ such that $(A - \lambda I_n)\underline{v} = \underline{0}$

$\iff$ $\text{Nul}(A - \lambda I_n) \neq \{\underline{0}\}$

$\iff$ Columns of $A - \lambda I_n$ L.D.

$\iff$ $(A - \lambda I_n)$ <u>not</u> invertible

$\iff$ $\det(A - \lambda I_n) = 0$

<u>Definition</u> The characteristic polynomial of A, an $n \times n$ matrix is given by $\det(A - xI_n)$, where x is a variable.

<u>Example</u> $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix} \Rightarrow A - xI_3 = \begin{pmatrix} 1-x & 3 & 3 \\ -3 & -5-x & -3 \\ 3 & 3 & 1-x \end{pmatrix}$

$$\Rightarrow \det(A - xI_3) = (1-x)\det\begin{pmatrix} -5-x & -3 \\ 3 & 1-x \end{pmatrix} - 3\det\begin{pmatrix} -3 & -3 \\ 3 & 1-x \end{pmatrix} + 3\det\begin{pmatrix} -3 & -5-x \\ 3 & 3 \end{pmatrix}$$

$$= (1-x)\big((-5-x)(1-x) - 3(-3)\big) - 3\big((-3)(1-x) - 3(-3)\big) + 3\big((-3)3 - 3(-5-x)\big)$$

$$= -x^3 - 3x^2 + 4$$

Facts: 1/ A - $n\times n$ matrix $\Rightarrow$ $\det(A - xI_n)$ is degree n polynomial

2/ λ an eigenvalue of A $\Leftrightarrow$ λ is a zero of characteristic polynomial. ie $\det(A - \lambda I_n) = 0$.

1/, 2/ $\Rightarrow$ A has at most n distinct eigenvalues.

Example

$$A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix} \Rightarrow \det(A - xI_3) = -x^3 - 3x^2 + 4 = -(x-1)(x+2)^2$$

$\Rightarrow$ $\{1, -2\}$ are eigenvalues of A

Definition λ - eigenvalue of A

Algebraic Multiplicity of λ := Number of times $(x - \lambda)$ divides $\det(A - xI)$

Example

For $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$, 1 has algebraic multiplicity 1 and -2 has algebraic multiplicity 2.

Warning: It is possible for A to have no real eigenvalues. E.g. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow \det(A - xI_2) = x^2 + 1$

Counterclockwise Rotation by $\frac{\pi}{2}$ around $\underline{0}$

<u>Theorem</u> The eigenvalues of an upper triangular matrix are the entries of the leading diagonal.

<u>Proof</u>

$$A = \begin{pmatrix} a_{11} & & & * \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix} \Rightarrow A - xI_n = \begin{pmatrix} (a_{11}-x) & & & * \\ & (a_{22}-x) & & \\ & & \ddots & \\ 0 & & & (a_{nn}-x) \end{pmatrix}$$

determinant of upper triangular matrix is product of diagonal entries

$$\Rightarrow \det(A - xI_n) = (a_{11}-x)(a_{22}-x)\dots(a_{nn}-x)$$

$\Rightarrow \{a_{11}, \dots, a_{nn}\}$ = zeros of characteristic polynomial

$\Rightarrow \{a_{11}, \dots, a_{nn}\}$ = eigenvalues of A. $\square$

<u>Warning</u>: Row reduction does not preserve characteristic polynomial.

$$A \xrightarrow{\text{Row operations}} \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$\not\Rightarrow \det(A - xI_n) = (\lambda_1 - x)\dots(\lambda_n - x)$$