Direct Products and Direct Sums

Let \( G, H \) be two groups.

**Definition.** The **direct product** of \( G \) and \( H \)
is the cartesian product \( G \times H = \{(g, h) \mid g \in G, h \in H\} \)
equipped with the binary operation
\[
(g_1, h_1) \ast (g_2, h_2) := (g_1 g_2, h_1 h_2)
\]
\( \forall g_1, g_2 \in G, h_1, h_2 \in H \)

**Proposition.** \( G \times H \) equipped with this composition is
a group.

**Proof.** Let \( g_1, g_2, g_3 \in G \) and \( h_1, h_2, h_3 \in H \)
\[
(g_1, h_1) \ast (g_2, h_2) \ast (g_3, h_3) = (g_1 g_2, h_1 h_2) \ast (g_3, h_3)
= (g_1 g_2 g_3, (h_1 h_2) h_3) = (g_1, (g_2 g_3, h_1 (h_2 h_3))
= (g_1, (h_1) \ast ((g_2, h_2) \ast (g_3, h_3))
= (g_1, h_1) \ast ((g_2, h_2) \ast (g_3, h_3))
\]
\[
(e_G, e_H) \ast (g, h) = (e_G g, e_H h) = (g, h)
\]
\[
(g, h) \ast (e_G, e_H) = (g e_G, h e_H) = (g, h)
\]
\( \forall g \in G \) and \( \forall h \in H \) \( (e_G, e_H) \) identity

1. \( (g, h) \ast (g^{-1}, h^{-1}) = (g g^{-1}, h h^{-1}) = (e_G, e_H) \)
2. \( (g, h) \ast (g^{-1}, h^{-1}) = (g g^{-1}, h h^{-1}) = (e_G, e_H) \)
3. \( (g, h) \ast (g^{-1}, h^{-1}) = (g g^{-1}, h h^{-1}) = (e_G, e_H) \)
\( \forall g \in G \) and \( \forall h \in H \) \( (g, h) \) is inverse to \( (g, h) \).

\( \square \)
Remarks

1. We can form the direct product of any collection of groups. As above it is just the cartesian product of them as sets with component by component composition.

2. For any $n \in \mathbb{N}$ we write
$$G^n = \underbrace{G \times G \times G \cdots \times G}_{n \text{ times}}$$

Familiar example: $(\mathbb{R}^n, +)$

3. $G \times H$ comes equipped with 2 "projection" homomorphisms

$$\phi_g : G \times H \rightarrow G \quad \phi_h : G \times H \rightarrow H$$

$$\phi_g(g, h) = g \quad \phi_h(g, h) = h$$

$$\ker \phi_g = \{(e_G, h) \mid h \in H\} \subseteq H$$
$$\ker \phi_h = \{(g, e_H) \mid g \in G\} \subseteq G$$

$\Rightarrow G$ and $H$ can be identified with subgroups of $G \times H$.

Familiar Example: Identifying $(\mathbb{R}, +)$ with $x$-axis or $y$-axis in $(\mathbb{R}^2, +)$.

4. $G \times H$ Abelian $\iff$ $G$ Abelian and $H$ Abelian.
Definition: Let $G$ be a group. Let $H, K \subseteq G$ be subgroups. We say that $G$ is a direct sum of $H$ and $K$ if
$$\forall h \in H \text{ and } k \in K \quad hk = kh$$
and given $g \in G$, $\exists h \in H$ and $k \in K$ s.t. $g = hk$.
We denote this by $G = H \oplus K$.

We can replace 2/ by
2'/ $H \cdot K = \{e \} \quad$ and given $g \in G$, $\exists h \in H$ and $k \in K$ s.t. $g = hk$.

Justification: $h, k_1 = h_2, k_2 \Rightarrow h^{-1} h_1 = k_2 k_1^{-1}$
\[ \in H \quad \in K \]
\[ \Rightarrow h_2^{-1} h_1, k_2 k_1^{-1} \in H \cdot K \]

Remarks: We can generalise this to any finite collection of subgroups in an obvious way:
$G = H_1 \oplus H_2 \oplus H_3 \cdots \oplus H_n$
\[ \iff \quad h_i h_j = h_j h_i \quad \forall h_i \in H_i, h_j \in H_j \quad (i \neq j) \]
and given $g \in G$, $\exists h_i \in H_i$ such that $g = h_1 h_2 \cdots h_n$.

2/ We can make sense at a direct sum of infinitely many subgroups. It is more subtle because remember we can only compose finitely many terms in $G$ at a time.
Theorem: Let $G$ be the direct sum of subgroups $H_1, \ldots, H_n \subset G$ then $G \cong H_1 \times H_2 \times \cdots \times H_n$.

Proof: We need to write down an isomorphism

$$\varphi : H_1 \times H_2 \times \cdots \times H_n \rightarrow G$$

Let $\varphi (h_1, h_2, \ldots, h_n) = h_1, h_2, \ldots, h_n$.

Property 1 of Direct Sum $\Rightarrow \varphi$ surjective.

$$h, g, \in H_1, h_2, g_2, \in H_2, \ldots, h_n, g_n, \in H_n$$

$$\varphi (h_1, h_2, \ldots, h_n) \ast (g_1, g_2, \ldots, g_n)$$

$$= \varphi (h_1, h_2, \ldots, h_n) \ast (g_2, g_2, \ldots, g_n)$$

$$= h_1, h_2, g_2, \ldots, h_n, g_n$$

$$= \varphi (h_1, h_2, \ldots, h_n) \varphi (g_2, g_2, \ldots, g_n)$$

Property 2 of Direct Sum $\Rightarrow \ker \varphi = \{e, e, e, \ldots, e\}$

$\Rightarrow \varphi$ injective.

$\square$