Bases of Eigenvectors and Diagonalization

A – $n \times n$ matrix

Q_1: Can we find a basis of $\mathbb{R}^n$ consisting of eigenvectors of A?
What consequences would this have?

Extended Application : Discrete Dynamic Systems

Discrete Dynamic System = System which changes at discrete time intervals $0, 1, 2, 3, \dots$

1) At each time k the state of the system can be described by a vector $\underline{x}_k$ in $\mathbb{R}^n$.

2) Each time transition is governed by a fixed $n \times n$ matrix A

ie $\underline{x}_{k+1} = A \underline{x}_k$

Important Question : What is the long term behavior of such a system?

Example Owl and rat population measured yearly

$$\underline{x}_k = \begin{pmatrix} a_k \\ b_k \end{pmatrix}, \quad \begin{array}{l} a_k = \text{population of owls at } t = k \\ b_k = \text{population of rats at } t = k \end{array}$$

$$\begin{pmatrix} a_{k+1} \\ b_{k+1} \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -3/2 & 7/2 \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix}$$

starting state

$$A = \begin{pmatrix} -1 & 3 \\ -3/2 & 7/2 \end{pmatrix}, \quad \underline{x}_0 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$\det(A - xI_2) = \frac{1}{2}(2x-1)(x-2) \Rightarrow \{(\frac{1}{2}), 2\}$ eigenvalues of A

$$A - \tfrac{1}{2} I_2 = \begin{pmatrix} -3/2 & 3 \\ -3/2 & 3 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -3/2 & 3 & 0 \\ -3/2 & 3 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \text{Nul}(A - \tfrac{1}{2} I_2) = \text{Span}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$$

$$A - 2I_2 = \begin{pmatrix} -3 & 3 \\ -3/2 & 3/2 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -3 & 3 & 0 \\ -3/2 & 3/2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow Nul(A - 2I_2) = Span\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

eigenvalue $\frac{1}{2}$

$$\Rightarrow \quad \left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} = \text{Basis of eigenvectors}$$

eigenvalue 2

$$\underline{x}_0 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\underline{x}_1 = A\begin{pmatrix} 3 \\ 2 \end{pmatrix} = A\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\underline{x}_2 = A^2\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \left(\frac{1}{2}\right)^2\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2^2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vdots$$

$$\underline{x}_k = A^k\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \left(\frac{1}{2}\right)^k\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2^k\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lim_{k\to\infty}\left(\frac{1}{2}\right)^k = 0 \quad \Rightarrow \quad \underline{x}_k \approx \begin{pmatrix} 2^k \\ 2^k \end{pmatrix} \text{ for } k \text{ very large.}$$

Awesome!

Other questions: Is there an $\underline{x}_0$ such that $\underbrace{\underline{x}_0 = \underline{x}_1 = \underline{x}_2 \cdots}_{\text{equilibrium}}$?

What $\underline{x}_0$ guarantees $\underline{x}_k \to \underline{0}$ as $k \to \infty$?

Back to theory:

Recall: 1/ $\{\underline{b}_1, \dots, \underline{b}_n\}$ a basis for $\mathbb{R}^n \iff P = (\underline{b}_1 \dots \underline{b}_n)$ invertible

2/ If so $P\underline{e}_i = \underline{b}_i$ and $P^{-1}\underline{b}_i = \underline{e}_i$

Theorem

$\{\underline{b}_1, \dots, \underline{b}_n\}$ a basis of eigenvectors $\iff P^{-1}AP$ diagonal.

Proof

($\Rightarrow$) Assume $A\underline{b}_i = \lambda_i \underline{b}_i$

$$\Rightarrow \quad P^{-1}AP\underline{e}_i = P^{-1}A\underline{b}_i = \lambda_i P^{-1}\underline{b}_i = \lambda_i \underline{e}_i$$

$\Rightarrow$ $\underline{e}_i$ eigenvector of $P^{-1}AP$ with eigenvalue λ_i

$$\Rightarrow \quad P^{-1}AP = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

($\Leftarrow$) Assume $P^{-1}AP = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

$$\Rightarrow \quad P^{-1}AP\underline{e}_i = \lambda_i \underline{e}_i \quad \Rightarrow \quad AP\underline{e}_i = \lambda_i P\underline{e}_i \Rightarrow A\underline{b}_i = \lambda_i \underline{b}_i$$

$\square$

Example $A = \begin{pmatrix} -1 & 3 \\ -3/2 & 7/2 \end{pmatrix}$, $P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

eigenvalue of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 3 \\ -3/2 & 7/2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$$

eigenvalue of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Definition A, B – $n \times n$ matrices are <u>similar</u> if

$P^{-1}AP = B$ for some invertible P.

Conclusion

There exists $\{\underline{b}_1, \dots, \underline{b}_n\}$ a basis of eigenvectors of A $\Leftrightarrow$ A similar to diagonal matrix D

If so we say A diagonalizable

Useful Consequence : $P^{-1}AP = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

$$\Rightarrow \quad (P^{-1}AP)^k = P^{-1}A\cancel{P}\cancel{P^{-1}}A\cancel{P} \dots AP = P^{-1}A^kP = D^k = \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix}$$

$$\Rightarrow \quad A^k = P \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix} P^{-1}$$

easy to calculate

Q, Is every square matrix diagonalizable?

No!

Problem 1 : May be no real eigenvalues.

Example: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow$ No real eigenvectors

Problem 2 : Dimension of eigenspaces may be too small.

Example : $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow 0$ only eigenvalue

$Nul(A - 0I_2) = Nul(A) = Nul\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \text{Span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$

One dimensional
Need a basis for $\mathbb{R}^2$

Theorem A – $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_p$.

1/ $\dim(\lambda_i\text{-eigenspace}) \leq$ Algebraic multiplicity of λ_i

$\Rightarrow \dim(\lambda_1\text{-eigenspace}) + \ldots + \dim(\lambda_p\text{-eigenspace}) \leq n$ degree of char. poly.

2/ A diagonalizable $\Leftrightarrow$

$\dim(\lambda_1\text{-eigenspace}) + \ldots + \dim(\lambda_p\text{-eigenspace}) = \dim(\mathbb{R}^n) = n$

3/ If this is the case and $\mathcal{B}_i$ is a basis for λ_i-eigenspace then $\mathcal{B}_1 \cup \mathcal{B}_2 \ldots \cup \mathcal{B}_p$ is a basis for $\mathbb{R}^n$.

Important Consequence : If A has exactly n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ then A diagonalizable.

Justification :

$\dim(\lambda_i\text{-eigenspace}) \geq 1$ for all n

$\Rightarrow \dim(\lambda_1\text{-eigenspace}) + \ldots + \dim(\lambda_n\text{-eigenspace}) \geq n$

$\overset{1/}{\Rightarrow} \dim(\lambda_1\text{-eigenspace}) + \ldots + \dim(\lambda_n\text{-eigenspace}) = n$

$\Rightarrow A$ diagonalizable

Conclusions

1) If A diagonalizable, to find basis of eigenvectors, find basis for $\mathrm{Null}(A-\lambda I_n)$ for each eigenvalue λ and take union.

2) If $\{\underline{b}_1, \ldots, \underline{b}_n\}$ is a basis of eigenvectors and $P = (\underline{b}_1, \ldots \underline{b}_n)$:

$$P^{-1}AP = \underbrace{D}_{\text{diagonal with eigenvalues on diagonal}} \quad \left(\text{and so } A = PDP^{-1}\right)$$

Example

$$A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix} \Rightarrow \det(A - xI_3) = -(x-1)(x+2)^2$$

$\Rightarrow \{\underset{\text{multiplicity } 1}{1}, \underset{\text{multiplicity } 2}{-2}\}$ are eigenvalues

$$A - 1 \cdot I_3 = \begin{pmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \mathrm{Nul}(A - 1 \cdot I_3) = \left\{ \begin{pmatrix} x_3 \\ -x_3 \\ x_3 \end{pmatrix} \right\}$$

$$= \mathrm{Span}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$A - (-2)I_3 = \begin{pmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \mathrm{Nul}(A - (-2)I_3) = \left\{ \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} \right\}$$

$$= \mathrm{Span}\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)$$

$\dim(1\text{-eigenspace}) + \dim((-2)\text{-eigenspace}) = 1 + 2 = 3$

$\Rightarrow$ A diagonalizable and $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis of eigenvectors

$$\Rightarrow \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$