Diagonalization of Symmetric Matrices

A – $n\times n$ matrix

Definition

A symmetric $\iff$ $A^T = A$ (ie $(A)_{ij} = (A)_{ji}$ for all i,j)

Examples $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ symmetric , $\begin{pmatrix} 2 & -1 & 2 \\ 1 & 3 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ non-symmetric

$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ symmetric

Q: What special properties do symmetric matrices have ?

Definition

A, an $n\times n$ matrix, is <u>orthogonally diagonalizable</u> if there exist an <u>orthonormal</u> basis $\{\underline{u}_1, \dots, \underline{u}_n\}$ for $\mathbb{R}^n$ consisting of <u>eigenvectors</u> of A.

For standard inner product on $\mathbb{R}^n$

Example $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ orthogonally diagonalizable with orthogonal basis of eigenvectors $\{\underline{e}_1, \dots, \underline{e}_n\}$

Recall :

$n\times n$ orthogonal matrix

$\{\underline{u}_1, \dots, \underline{u}_n\} \subset \mathbb{R}^n$ an orthonormal basis $\iff$ $P = (\underline{u}_1 \dots \underline{u}_n)$ invertible with $P^{-1} = P^T$.

Conclusion :

A <u>orthogonally</u> diagonalizable $\iff$ There exists an orthogonal matrix P such that $P^T A P$ diagonal.

Theorem A orthogonally diagonalizable $\Rightarrow$ A symmetric

Proof Let $\{\underline{u}_1,\dots,\underline{u}_n\} \subset \mathbb{R}^n$ be an orthonormal basis such that $A\underline{u}_i = \lambda_i \underline{u}_i$ for some λ_i in $\mathbb{R}$. Let $P = (\underline{u}_1 \dots \underline{u}_n)$

$$\Rightarrow \quad P^T A P = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \Rightarrow A = P \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} P^T$$

P orthogonal matrix

$\left((BC)^T = C^T B^T \right)$

$$\Rightarrow A^T = \left(P \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} P^T \right)^T = (P^T)^T \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}^T P^T = P \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} P^T = A$$

$\Rightarrow$ A symmetric

$\square$

Spectral Theorem for Symmetric Matrices

A symmetric $\Rightarrow$ A orthogonally diagonalizable

Highly non-obvious. Not easy to prove

Consequences :

1) All zeros of $\det(A - x I_n)$ are real.

2) Eigenvectors from different eigenspaces are orthogonal.

3) $\dim(\lambda\text{-eigenspace})$ = Algebraic Multiplicity of λ for all eigenvalues

To find an orthonormal basis go through following steps

a) Calculate $\det(A - x I_n)$ and find all zeroes $\lambda_1,\dots,\lambda_p$.

b) Do row reduction to find basis for $\mathrm{Nul}(A - \lambda_i I_n) = \lambda_i$-eigenspace

c) Apply Gram-Schmidt to each basis and normalize before taking union

<u>Example</u> $\quad A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

a) $\det(A - xI_3) = \begin{vmatrix} 1-x & 1 & 1 \\ 1 & 1-x & 1 \\ 1 & 1 & 1-x \end{vmatrix} = -x^2(x-3)$

$\Rightarrow$ 0, 3 are eigenvalues

b) $Nul(A - 0I_3) = Nul\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow Nul(A - 0I_3) = \left\{ \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} \right\}$$

$$\Rightarrow Nul(A - 2I_3) = Span\underbrace{\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)}_{\text{Basis}}$$

$Nul(A - 3I_3) = Nul\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow Nul(A - 3I_3) = \left\{ \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix} \right\}$$

$$= Span\Bigg(\underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\text{Basis}}\Bigg)$$

c)

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \xrightarrow{G-S} \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix} \right\} \xrightarrow{\text{Normalize}} \underbrace{\left\{ \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} \right\}}_{\text{orthonormal basis for 0-eigenspace}}$$

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \xrightarrow{G-S.} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \xrightarrow{\text{Normalize}} \left\{ \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \right\}$$

Let $P = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{pmatrix} \Rightarrow P^T A P = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Useful Consequence of Spectral Theorem : Spectral Decompositions

A symmetric $\Rightarrow$ $A = P \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} P^T$ ($P = (\underline{u}_1 \dots \underline{u}_n)$ orthogonal matrix)

$\Rightarrow \quad A = \lambda_1 \underline{u}_1 \underline{u}_1^T + \lambda_2 \underline{u}_2 \underline{u}_2^T + \dots + \lambda_n \underline{u}_n \underline{u}_n^T$

Called a Spectral Decomposition

Recall : $(\underline{u}_i \underline{u}_i^T)\underline{v} = \mathrm{Proj}_{\mathrm{span}(\underline{u}_i)}(\underline{v})$

$\Rightarrow \quad A\underline{v} = \lambda_1 \mathrm{Proj}_{\mathrm{span}(\underline{u}_1)}(\underline{v}) + \dots + \lambda_n \mathrm{Proj}_{\mathrm{span}(\underline{u}_n)}(\underline{v})$

<u>Picture</u>