The Determinant

called determinant of A, $\det(A)$

Recall $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ invertible $\iff$ $ad - bc \neq 0$

Q: Does this concept extend to $n \times n$ matrices for $n > 2$?

A – $n \times n$ matrix.

For $1 \le i, j \le n$ let A be $(n-1) \times (n-1)$ matrix formed by removing i^{th} row and j^{th} column.

Example $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \Rightarrow A_{22} = \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix}$

$n = 3$ $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

We define

$$\det(A) = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13})$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31})$$

$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Terrible Formula

Observe we used the 2×2 determinant to define the 3×3. The general case follows that pattern.

Definition A - $n \times n$ matrix

$(n-1) \times (n-1)$ determinant

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \dots + (-1)^{1+n} a_{1n}\det(A_{1n})$$

We often write $|A| = \det(A)$

Facts

1/ We can calculate $\det(A)$ using this alternating sum using any row or even column. The following pattern must be followed:

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \cdots \\ \vdots & & \vdots & \end{pmatrix}$$

Example $A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$. Calculate $\det A$ using 1st column.

$$\begin{vmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} - 0\begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= 2(2 \cdot 3 - 0 \cdot 1) = 2 \cdot 2 \cdot 3 = 12$$

Definition An $n \times n$ matrix is upper triangular if it has all zero entries below the diagonal.

Example $\begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ diagonal.

Fact : A upper triangular $\Rightarrow$ $\det(A) = a_{11}a_{22}a_{33}\cdots a_{nn}$

Show by repeatedly calculating determinant with 1st column

Product of entries on diagonal.

Note : A $n\times n$ matrix in echelon form $\Rightarrow$ A upper triangular

Very Useful Fact :

A/ Switching two rows in a matrix multiplies the determinant by -1

B/ Adding a scalar multiple of one row to another does not change determinant.

Algorithm to compute $\det A$:

1/ Put A in echelon form using only A/ and B/

2/ Compute product of diagonal entries (This is the determinant of triangular matrix)

3/ $\det(A) = (-1)^r \times$ Product of diagonal entries in echelon form

Number of row switches

Example

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 3 \\ 1 & 0 & 3 & 1 \\ 0 & 2 & 3 & 1 \end{vmatrix} = ?$$

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 3 \\ 1 & 0 & 3 & 1 \\ 0 & 2 & 3 & 1 \end{pmatrix} \xrightarrow{\text{Switch } 1^{st} \text{ and } 2^{nd}} \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 3 & 1 \\ 0 & 2 & 3 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & 2 & 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\Longrightarrow \begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 3 \\ 1 & 0 & 3 & 1 \\ 0 & 2 & 3 & 1 \end{vmatrix} = (-1) \cdot \underbrace{1 \cdot 1 \cdot 1 \cdot 2}_{=\,-2}$$

Conclusion :

$$\boxed{\begin{aligned} \det(A) \neq 0 &\iff A \text{ row equivalent to } \begin{pmatrix} \blacksquare & * & * & \cdots & * \\ 0 & \blacksquare & * & & \\ 0 & 0 & \blacksquare & & \\ \vdots & & & \ddots & \\ & & & & \blacksquare\ * \\ 0 & \cdots & & 0 & \blacksquare \end{pmatrix} \\ &\iff A \text{ row equivalent to } I_n \\ &\iff A \ \underline{\text{invertible}} \end{aligned}}$$

Other Properties of det :

1/ $\det(AB) = \det(A)\det(B)$

2/ $\det(A^T) = \det(A)$