Constant Coefficient Linear Systems of Differential Equations

$$\underline{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \qquad A = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$n \times n$ matrix with constant real entries

Aim: Find general solution to $\underline{x}'(t) = A\underline{x}(t)$ on $(-\infty, \infty)$

Observation: If $\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ in $\mathbb{R}^n$ is an eigenvector of A with eigenvalue λ, then $\underline{x}(t) = e^{\lambda t}\underline{v} = \begin{pmatrix} e^{\lambda t} v_1 \\ \vdots \\ e^{\lambda t} v_n \end{pmatrix}$ is a solution.

$$\left(\underline{x}'(t) = \begin{pmatrix} \lambda e^{\lambda t} v_1 \\ \vdots \\ \lambda e^{\lambda t} v_n \end{pmatrix} = e^{\lambda t} \lambda \underline{v} = e^{\lambda t} A \underline{v} = A(e^{\lambda t}\underline{v}) = A\underline{x}(t) \right)$$

Theorem If $\{\underline{v}_1, \dots, \underline{v}_n\} \subset \mathbb{R}^n$ is a basis of eigenvectors of A with eigenvalues $\lambda_1, \dots, \lambda_n$ then

$$\underline{x}_1(t) = e^{\lambda_1 t}\underline{v}_1, \ \underline{x}_2(t) = e^{\lambda_2 t}\underline{v}_2, \dots, \ \underline{x}_n(t) = e^{\lambda_n t}\underline{v}_n$$

is a fundamental solution set.

Proof

$\{\underline{x}_1, \dots, \underline{x}_n\}$ fundamental solution set $\iff W[\underline{x}_1, \dots, \underline{x}_n](t) \neq 0$ for all t in $(-\infty, \infty)$

$$\begin{aligned} W[\underline{x}_1, \dots, \underline{x}_n](t) &:= \det\left(\underline{x}_1(t) \ \cdots \ \underline{x}_n(t)\right) \\ &= \det\left(e^{\lambda_1 t}\underline{v}_1 \ \cdots \ e^{\lambda_n t}\underline{v}_n\right) \\ &= e^{\lambda_1 t} e^{\lambda_2 t} \cdots e^{\lambda_n t} \underbrace{\det(\underline{v}_1 \cdots \underline{v}_n)}_{\neq 0} \\ &\neq 0 \text{ for all } t \text{ in } (-\infty, \infty) \end{aligned}$$

Multiply column by α multiplies determinant by α

$\{\underline{v}_1, \dots, \underline{v}_n\}$ a basis for $\mathbb{R}^n$

$\square$

Conclusion: If A diagonalizable we can completely solve
$x'(t) = A\underline{x}(t)$.

Example $\underline{x}'(t) = \overset{= A}{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}} \underline{x}(t)$

$$\det(A - x I_2) = \det\begin{pmatrix} 2-x & 1 \\ 1 & 2-x \end{pmatrix} = (2-x)^2 - 1 = \underset{\overset{\shortparallel}{(x-3)(x-1)}}{x^2 - 4x + 3} = 0$$

$\Rightarrow$ eigenvalues are 1 and 3

$$\text{Nul}(A - 1 . I_2) = \text{Nul}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \text{Span}\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$$

$$\text{Nul}(A - 3 I_2) = \text{Nul}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \text{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$\Rightarrow$ $\left\{ e^{t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ a fundamental solution set.

Q: What happens if there are non-real roots to characteristic equation?

Example: $x'(t) = \overset{= A}{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \underline{x}(t)$

$\Rightarrow$ $\det(A - x I_2) = \det\begin{pmatrix} -x & 1 \\ 1 & -x \end{pmatrix} = x^2 + 1 = 0$

$\Rightarrow$ $x = \pm i$ $\leftarrow$ non-real eigenvalues

Remark: We can still apply same tools of Linear algebra when dealing with complex numbers.

Example

$$\text{Nul}(A - i I_2) = \text{Nul}\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}$$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}$$

$i(-i) - 1 = -(-1) - 1 = 0$

$$\text{Nul}\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} = \left\{ \begin{pmatrix} i x_2 \\ x_2 \end{pmatrix} \right\} = \text{Span}\left(\begin{pmatrix} i \\ 1 \end{pmatrix} \right)$$

Cool Fact: $\underline{x}(t) = e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix}$ is still a solution to $\underline{x}'(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \underline{x}(t)$

$$\left(\underline{x}'(t) = i e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix} = e^{it}\, i \begin{pmatrix} i \\ 1 \end{pmatrix} = e^{it} A \begin{pmatrix} i \\ 1 \end{pmatrix} = A\left(e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix}\right) = A\underline{x}(t) \right)$$

Problem: We want real solutions.

Observation:

$$e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix} = e^{it}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

$$= (\cos(t) + i\sin(t))\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

$$= \left(\cos(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sin(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

$$+ i\left(\cos(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

Fact:

$$\underline{x}_1(t) = \cos(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sin\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$$

$$\underline{x}_2(t) = \cos(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$

are fundamental (real) solution set.

Remark Going through same process with $-i$ instead will produce scalar multiples of same solution set.

General Situation:

A – $n \times n$ real matrix with $\alpha + i\beta$ a non-real eigenvalue with eigenvector $\underline{a} + i\underline{b}$ ($\underline{a}, \underline{b}$ in $\mathbb{R}^n$) then

$$\underline{x}_1(t) = e^{\alpha t}\cos\beta t\, \underline{a} - e^{\alpha t}\sin(\beta t)\, \underline{b}$$

$$\underline{x}_2(t) = e^{\alpha t}\sin\beta t\, \underline{a} + e^{\alpha t}\cos(\beta t)\, \underline{b}$$

are (real) L.I. solutions to $\underline{x}'(t) = A\,\underline{x}(t)$

Example

Find $\underline{x}(t)$ such that $\underline{x}'(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\underline{x}(t)$ and $\underline{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\left\{ \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}, \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \right\}$ Fundamental Solution set

$$\Rightarrow \quad \underline{x}(t) = c_1 \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} + c_2 \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$

$$\underline{x}(0) = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow c_1 = c_2 = 1$$

$$\Rightarrow \quad \underline{x}(t) = \begin{pmatrix} \cos(t) - \sin(t) \\ \cos(t) + \sin(t) \end{pmatrix}$$