Bases and Coordinate Systems

V - finite dimensional vector space

$\mathcal{B} = \{\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n\}$ basis for V

Aim: Use $\{\underline{b}_1, \ldots, \underline{b}_n\}$ to think about V in a more concrete way

Spanning

Important Fact: Each $\underline{x}$ in V can be written as a linear combination $\underline{x} = \lambda_1 \underline{b}_1 + \cdots + \lambda_n \underline{b}_n$ in a unique way

Linearly independent

Definition: The coordinate vector of $\underline{x}$ relative to the basis $\{\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n\}$ is $\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$ in $\mathbb{R}^n$, where $\underline{x} = \lambda_1 \underline{b}_1 + \cdots + \lambda_n \underline{b}_n$.

Notation: $[\underline{x}]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$

Examples 1) $V = \mathbb{R}^n$, $\mathcal{B} = \mathcal{E} = \{\underline{e}_1, \ldots, \underline{e}_n\}$, the standard basis

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \underline{e}_1 + x_2 \underline{e}_2 + \cdots + x_n \underline{e}_n \Rightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{\mathcal{E}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$\Rightarrow$ In $\mathbb{R}^n$ Standard Basis Coordinates are usual coordinate representation.

2) $\mathbb{P}_n(\mathbb{R})$, $\mathcal{B} = \{1, x, x^2, \ldots, x^n\}$ Standard Basis for $\mathbb{P}_n(\mathbb{R})$

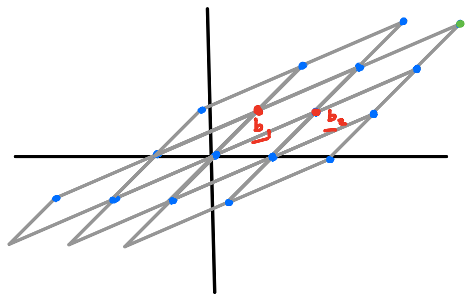
$$\left(a_0 + a_1 x + \cdots + a_n x^n\right)_{\mathcal{B}} = \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix}$$

3) $V = \mathbb{R}^2$, $\mathcal{B} = \Big\{ \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\underline{b}_1}, \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{\underline{b}_2} \Big\}$, $\begin{pmatrix} 5 \\ 3 \end{pmatrix}_{\mathcal{B}} = ?$

$$\left(\begin{array}{cc|c} 1 & 2 & 5 \\ 1 & 1 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -1 & -2 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right)$$

$\begin{pmatrix} 5 \\ 3 \end{pmatrix}_{\mathcal{B}}$

$$\Rightarrow \begin{pmatrix} 5 \\ 3 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 5 \\ 3 \end{pmatrix}_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$



$$\begin{pmatrix}5\\3\end{pmatrix} = 1\cdot\begin{pmatrix}1\\1\end{pmatrix} + 2\begin{pmatrix}2\\1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}5\\3\end{pmatrix}_{\mathcal{B}} = \begin{pmatrix}1\\2\end{pmatrix}$$

<u>General Situation in $\mathbb{R}^n$</u> : $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ basis for $\mathbb{R}^n$

$$\Rightarrow \left(\underline{b}_1 \dots \underline{b}_n \,\middle|\, \underline{x}\right) \xrightarrow{\text{Row Operations}} \left(I_n \,\middle|\, (\underline{x})_{\mathcal{B}}\right)$$

<u>Key Observation</u> : Choosing a basis $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ for V allows us to identify V with $\mathbb{R}^n$

Linear, one-to-one, onto

$$(\ )_{\mathcal{B}} : V \longrightarrow \mathbb{R}^n \qquad \underline{b}_i = 0\cdot\underline{b}_1 + \dots + 1\cdot\underline{b}_i + \dots + 0\,\underline{b}_n \Rightarrow (\underline{b}_i)_{\mathcal{B}} = \underline{e}_i$$

$$\underline{x} \longmapsto (\underline{x})_{\mathcal{B}} \qquad \{\underline{b}_1, \dots, \underline{b}_n\} \longleftrightarrow \{\underline{e}_1, \dots, \underline{e}_n\}$$

Structural properties / vector calculations are preserved when switching from V to $\mathbb{R}^n$.

<u>Example</u>

Is $\{x^2+2x+1, \ x^2, \ x+1\}$ linearly independent in $\mathbb{P}_2(X)$?

$$\{x^2+2x+1, \ x^2, \ x+1\} \xleftrightarrow[\text{Basis}]{\text{Standard}} \text{-coordinates} \left\{ \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\}$$

Row Reduction $\Rightarrow \left\{ \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\}$ L.I. in $\mathbb{R}^3$

$\Rightarrow \{x^2+2x+1, \ x^2, \ x+1\}$ L.I. in $\mathbb{P}_2(\mathbb{R})$

Warning: Different bases give different coordinate systems.

<u>Example</u> $\quad V = \mathbb{R}^2, \quad \mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}, \quad C = \left\{ \begin{pmatrix} -1 \\ -6 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \end{pmatrix} \right\}$

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & 1 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right), \qquad \left(\begin{array}{cc|c} -1 & -1 & 3 \\ -6 & -2 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -4 \end{array}\right)$$

$$\Rightarrow \quad \begin{pmatrix} 3 \\ 2 \end{pmatrix}_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \underline{\text{and}} \quad \begin{pmatrix} 3 \\ 2 \end{pmatrix}_{\mathcal{B}} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

Q. How can we switch between coordinate systems?

$\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$, $C = \{\underline{c}_1, \dots, \underline{c}_n\}$ bases for V

$$\begin{tikzcd} & V \arrow[dl, "\mathcal{B}\text{-coordinates}"'] \arrow[dr, "C\text{-coordinates}"] & \\ \mathbb{R}^n \arrow[rr, "T = ?"'] & & \mathbb{R}^n \end{tikzcd}$$

$$\underline{x}, \qquad (\underline{x})_{\mathcal{B}} \longmapsto (\underline{x})_C$$

We need $T((\underline{x})_{\mathcal{B}}) = (\underline{x})_C$ for all $\underline{x}$ in V

<u>Fact</u>: T is linear, so $\quad A(\underline{x})_{\mathcal{B}} = (\underline{x})_C$ where A standard matrix of T

Recall $\quad \underline{b}_i = 0 \cdot \underline{b}_1 + \dots + 1 \cdot \underline{b}_i + \dots + 0\underline{b}_n \Rightarrow (\underline{b}_i)_{\mathcal{B}} = \underline{e}_i$

$\Rightarrow \quad T(\underline{e}_i) = T((\underline{b}_i)_{\mathcal{B}}) = (\underline{b}_i)_C \longleftarrow$ $\underline{b}_i$ written in C-coordinates

<u>Definition</u>: $\quad P_{C \leftarrow \mathcal{B}} = \left((\underline{b}_1)_C \; (\underline{b}_2)_C \; \cdots \; (\underline{b}_n)_C \right)$

Change of coordinate matrix from $\mathcal{B}$ to C

Key Property: $\quad P_{C \leftarrow \mathcal{B}} (\underline{x})_{\mathcal{B}} = (\underline{x})_C$ for all $\underline{x}$ in V

General Change of Basis in $\mathbb{R}^n$:

$$\left(\underline{c}_1 \cdots \underline{c}_n \mid \underline{b}_i \right) \longrightarrow \left(I_n \mid (\underline{b}_i)_c \right)$$

$$\Rightarrow \quad \left(\underline{c}_1 \cdots \underline{c}_n \mid \underline{b}_1 \cdots \underline{b}_n \right) \longrightarrow \left(I_n \mid P_{C \leftarrow B} \right)$$

Example $V = \mathbb{R}^2$, $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$, $C = \left\{ \begin{pmatrix} -1 \\ -6 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \end{pmatrix} \right\}$

$$\left(\begin{array}{cc|cc} -1 & -1 & 1 & 2 \\ -6 & -2 & 1 & 1 \\ \shortparallel & \shortparallel & \shortparallel & \shortparallel \\ \underline{c}_1 & \underline{c}_2 & \underline{b}_1 & \underline{b}_2 \end{array} \right) \xrightarrow[\text{operations}]{\text{Row}} \left(\begin{array}{cc|cc} 1 & 0 & 1/4 & 3/4 \\ 0 & 1 & -5/4 & -11/4 \end{array} \right)$$

$$\underbrace{\begin{pmatrix} 1/4 & 3/4 \\ -5/4 & -11/4 \end{pmatrix}}_{P_{C \leftarrow \mathcal{B}}}$$

$$\underbrace{\begin{pmatrix} 1/4 & 3/4 \\ -5/4 & -11/4 \end{pmatrix}}_{= P_{C \leftarrow \mathcal{B}}} \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{= \begin{pmatrix} 3 \\ 2 \end{pmatrix}_{\mathcal{B}}} = \underbrace{\begin{pmatrix} 1 \\ -4 \end{pmatrix}}_{= \begin{pmatrix} 3 \\ 2 \end{pmatrix}_{C}}$$