Abstract Algebra 113

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Question Sheet 3

Question 1

Let \((G, \ast)\) be a group and \(S\) a set. Let \(\varphi : G \rightarrow \Sigma(S)\) be a group homomorphism. Prove that the following map of sets:

\[ \mu : G \times S \rightarrow S \]
\[ (g, s) \rightarrow \varphi(g)(s) \]

is an action. This shows the equivalence between the two concepts of a group action.

Question 2

Prove that all cyclic groups are abelian. Hint: think about what an arbitrary element of a cyclic group looks like.

Question 3

Let \(n \in \mathbb{N}\). Write down a cyclic subgroup of \(\text{Sym}_n\) of order \(n\). Is it unique?

Question 4

Let \(G\) and \(H\) be two non-trival finite groups. Assume that there exists \(x, y \in \mathbb{Z}\) such that \(x|G| + y|H| = 1\). Prove that \(H\) is not isomorphic to a subgroup of \(G\).

Question 5

Let \(G\) be a finite group with 20 elements. Let \(S\) be a set with 15 elements. Does there exist a transitive action of \(G\) on \(S\)?

Question 6 (Hard)

Recall that in HW2 we showed that if \(p\) is a prime number then \((\mathbb{Z}/p\mathbb{Z}\setminus\{0\}, \times)\) is a group of order \(p - 1\). Using Lagranges Theorem show:

**Fermat’s Little Theorem.** Let \(p\) be a prime and \(a \in \mathbb{Z}\) coprime to \(p\). Then

\[ a^{p-1} \equiv 1 \mod(p). \]