MATH 54 MIDTERM 2 (PRACTICE 3)

PROFESSOR PAULIN

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CALCULATORS ARE NOT PERMITTED
YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK
SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE
REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT
THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name and Student ID: ______________________________

GSI's name: ______________________________

This exam consists of 5 questions. Answer the questions in the spaces provided.

1. (25 points) (a) Find a basis for the subspace of all vectors $\begin{pmatrix} a-2b+5c \\ 2a+5b-8c \\ -a-4b+7c \\ 3a+b+c \end{pmatrix}$ in $\mathbb{R}^4$.

Solution:

$$\left\{ \begin{pmatrix} a-2b+5c \\ 2a+5b-8c \\ -a-4b+7c \\ 3a+b+c \end{pmatrix} \right\} = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 5 \\ -4 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ -8 \\ 7 \\ 1 \end{pmatrix} \right)$$

$$\begin{pmatrix} 1 & -2 & 5 \\ 2 & 5 & -8 \\ -1 & -4 & 7 \\ 3 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 5 \\ 0 & 9 & -18 \\ 0 & -6 & 12 \\ 0 & 7 & -14 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & -2 & 5 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 5 \\ -4 \\ 1 \end{pmatrix} \right\} \text{ is a basis}$$

(b) What dimension is this subspace?

Solution:

The basis has size 2 so the dimension is 2.

2. (a) Let $A = (\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, \underline{\mathbf{a}}_3)$ be a 4×3 matrix with $A^T A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ 3 & -1 & 9 \end{pmatrix}$. Calculate $||\underline{\mathbf{a}}_1 + \underline{\mathbf{a}}_2 + \underline{\mathbf{a}}_3||$ and

Solution:

$$(A^T A)_{ij} = \underline{a}_i \cdot \underline{a}_j$$

$$||\underline{a}_1 + \underline{a}_2 + \underline{a}_3||^2 = (\underline{a}_1 + \underline{a}_2 + \underline{a}_3) \cdot (\underline{a}_1 + \underline{a}_2 + \underline{a}_3)$$

$$= \underline{a}_1 \cdot \underline{a}_1 + \underline{a}_2 \cdot \underline{a}_2 + \underline{a}_3 \cdot \underline{a}_3 + 2\underline{a}_1 \cdot \underline{a}_2 + 2\underline{a}_1 \cdot \underline{a}_3 + 2\underline{a}_2 \cdot \underline{a}_3$$

$$= 1 + 4 + 9 + 2 \times 2 + 2 \times 3 + 2 \times (-1)$$

$$= 22$$

$$\Rightarrow ||\underline{a}_1 + \underline{a}_2 + \underline{a}_3|| = \sqrt{22}$$

(b) Does there exist a 4×3 matrix B such that $B^T B = \begin{pmatrix} 2 & -1 & 1 \\ -1 & -1 & 3 \\ 1 & -1 & 9 \end{pmatrix}$. Justify your answer.

No. For two reasons.

1/ $\underline{b}_2 \cdot \underline{b}_2 = -1 < 0$ This is impossible.

2/ $\underline{b}_2 \cdot \underline{b}_3 = 3 \neq -1 = \underline{b}_3 \cdot \underline{b}_2$ This is impossible.

3. (25 points) Let V be a vector space with basis $B = \{\underline{\mathbf{b}}_1, \underline{\mathbf{b}}_2, \underline{\mathbf{b}}_3\}$. Let T be the linear transformation from V to V such that

$$T(\underline{\mathbf{b}}_1) = 4\underline{\mathbf{b}}_1 + \underline{\mathbf{b}}_3,\ T(\underline{\mathbf{b}}_2) = 4\underline{\mathbf{b}}_2 + \underline{\mathbf{b}}_3\ T(\underline{\mathbf{b}}_3) = -2\underline{\mathbf{b}}_3.$$

Find all possible λ such that there exists non-zero $\underline{\mathbf{v}}$ in V such that $T(\underline{\mathbf{v}}) = \lambda\underline{\mathbf{v}}$.

Solution:

$$A_{\beta,\beta} = \left((T(\underline{b}_1))_\beta \ (T(\underline{b}_2))_\beta \ (T(\underline{b}_3))_\beta \right)$$

$$= \left((4\underline{b}_1 + \underline{b}_3)_\beta \ (4\underline{b}_2 + \underline{b}_3)_\beta \ (-2\underline{b}_3)_\beta \right)$$

$$= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 1 & -2 \end{pmatrix}$$

$\underline{v} \neq \underline{0}$ $\qquad\qquad (\underline{v})_\beta \neq \underline{0}$

$T(\underline{v}) = \lambda \underline{v} \iff A_{\beta,\beta}(\underline{v})_\beta = \lambda(\underline{v})_\beta \iff \lambda$ an eigenvalue of $A_{\beta\beta}$

$$\det(A_{\beta,\beta} - x I_3) = \det \begin{pmatrix} 4-x & 0 & 0 \\ 0 & 4-x & 0 \\ 1 & 1 & -2-x \end{pmatrix}$$

$$= (4-x)(4-x)(-2-x)$$

$\Rightarrow$ $T(\underline{v}) = \lambda \underline{v}$ has non-zero solution $\iff \lambda = 4$ or -2.

4. Let $A = \begin{pmatrix} 7 & 4 \\ -3 & -1 \end{pmatrix}$. Given an explicit description of A^k for any positive integer k.

Solution:

$\det(A - xI_2) = (7-x)(-1-x) + 12 = x^2 - 6x + 5 = (x-1)(x-5)$

$\Rightarrow$ 1 and 5 are only eigenvalues

$A - 5I_2 = \begin{pmatrix} 2 & 4 \\ -3 & -6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \mathrm{Nul}(A - 5I_2) = \left\{ \begin{pmatrix} -2x_2 \\ x_2 \end{pmatrix} \right\} = \mathrm{Span}\left(\begin{pmatrix} -2 \\ 1 \end{pmatrix} \right)$

$A - 1 \cdot I_2 = \begin{pmatrix} 6 & 4 \\ -3 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \mathrm{Nul}(A - 1 \cdot I_2) = \left\{ \begin{pmatrix} -\frac{2}{3}x_2 \\ x_2 \end{pmatrix} \right\} = \mathrm{Span}\left(\begin{pmatrix} -2/3 \\ 1 \end{pmatrix} \right)$

$= \mathrm{Span}\left(\begin{pmatrix} -2 \\ 3 \end{pmatrix} \right)$

$\Rightarrow \begin{pmatrix} -2 & -2 \\ 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 7 & 4 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} -2 & -2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 7 & 4 \\ -3 & -1 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & -2 \\ 1 & 3 \end{pmatrix}^{-1}$

$\Rightarrow \begin{pmatrix} 7 & 4 \\ -3 & -1 \end{pmatrix}^k = \begin{pmatrix} -2 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 5^k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & -2 \\ 1 & 3 \end{pmatrix}^{-1}$

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5. (25 points) (a) Let $\mathbb{P}_2(\mathbb{R})$ be the vector space of polynomials of degree at most 2 with real coefficients. Let $B = \{1+x, 1, x^2-1\}$ and $C = \{1+x+x^2, 2x, 1\}$. Calculate the matrix $P_{C\leftarrow B}$.

Solution:

$$P_{C\leftarrow B} = \left((\underline{b}_1)_C \; (\underline{b}_2)_C \; (\underline{b}_3)_C \right) = \left((1+x)_C \; (1)_C \; (x^2-1)_C \right)$$

$$1+x = 0(1+x+x^2) + \tfrac{1}{2}(2x) + 1\cdot 1 \Rightarrow (1+x)_C = \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix}$$

$$1 = 0(1+x+x^2) + 0\cdot(2x) + 1\cdot 1 \Rightarrow (1)_C = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$x^2-1 = 1(1+x+x^2) + (-1/2)\cdot(2x) + (-2)\cdot 1 \Rightarrow (x^2-1)_C = \begin{pmatrix} 1 \\ -1/2 \\ -2 \end{pmatrix}$$

$$\Rightarrow P_{C\leftarrow B} = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 0 & -1/2 \\ 1 & 1 & -2 \end{pmatrix}$$

(b) Does there exist a non-zero polynomial $p(x)$ such that $(p(x))_B = (p(x))_C$?

Solution:

$p(x)$ non-zero $\qquad\qquad (p(x))_B$ non-zero

$$(p(x))_B = (p(x))_C \Leftrightarrow P_{C\leftarrow B}\,(p(\underline{x}))_B = (p(x))_B \Leftrightarrow$$ 1 is an eigenvalue of $P_{C\leftarrow B}$

$$\begin{vmatrix} -1 & 0 & 1 \\ 1/2 & -1 & -1/2 \\ 1 & 1 & -3 \end{vmatrix} = -1\left((-1)(-3) + \tfrac{1}{2}\right) + 1\left(1/2 + 1\right)$$

$$= \frac{-7}{2} + \frac{3}{2} = -2 \neq 0$$

$\Rightarrow$ No there does not exist such a $p(x)$.

END OF EXAM