MATH 54 MIDTERM 2 (PRACTICE 2)

PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.
CALCULATORS ARE NOT PERMITTED
YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK
SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE
REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT
THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name and Student ID: ______________________________

GSI's name: ______________________________

This exam consists of 5 questions. Answer the questions in the spaces provided.

1. (25 points) (a) Let $\mathbb{P}_3(\mathbb{R})$ be the vector space of polynomials of degree at most 3 with real coefficients. Calculate the dimension of the subspace

$$U = \text{Span}(1 + x - x^2, 2 + x^2 + x^3, 5 - 2x - x^3, 4 - 3x + x^2 - x^3\}$$

Solution:

standard coordinates

$$\{1+x-x^2,\ 2+x^2+x^3,\ 5-2x-x^3,\ 4-3x+x^2-x^3\} \longleftrightarrow \left\{\begin{pmatrix}1\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\\1\end{pmatrix}, \begin{pmatrix}5\\-2\\0\\-1\end{pmatrix}, \begin{pmatrix}4\\-3\\1\\-1\end{pmatrix}\right\}$$

$$\begin{pmatrix}1&2&5&4\\1&0&-2&-3\\-1&1&0&1\\0&1&-1&-1\end{pmatrix} \to \begin{pmatrix}1&2&5&4\\0&-2&-7&-7\\0&3&5&5\\0&1&-1&-1\end{pmatrix} \to \begin{pmatrix}1&2&5&4\\0&1&-1&-1\\0&3&5&5\\0&-2&-7&-7\end{pmatrix} \to \begin{pmatrix}\boxed{1}&2&5&4\\0&\boxed{1}&-1&-1\\0&0&\boxed{8}&8\\0&0&0&0\end{pmatrix}$$

$$\Rightarrow \dim\left(\text{Span}\left\{\begin{pmatrix}1\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\\1\end{pmatrix}, \begin{pmatrix}5\\-2\\0\\-1\end{pmatrix}, \begin{pmatrix}4\\-3\\1\\-1\end{pmatrix}\right\}\right) = 3$$

$$\Rightarrow \dim\left(\text{Span}\{1+x-x^2,\ 2+x^2+x^3,\ 5-2x-x^3,\ 4-3x+x^2-x^3\}\right) = 3$$

(b) Is $U = \mathbb{P}_3(\mathbb{R})$? Justify your answer.

Solution:

$$\dim(\mathbb{P}_3(\mathbb{R})) = 4, \quad \dim(U) = 3 \quad \Rightarrow \quad U \neq \mathbb{P}_3(\mathbb{R})$$

2. (25 points) (a) You are given a linear system with 5 equations in 6 unknowns. If the corresponding homogeneous linear system has a solution set spanned by two linearly independent vectors, is it true that the original linear system is guaranteed to have a solution? If it is not possible, give an explicit example of such a system.

Solution:

A - 5×6 matrix

$\dim(\mathrm{Nul}(A)) = 2$

$6 = \dim(\mathrm{Nul}(A)) + \dim(\mathrm{Col}(A)) \Rightarrow \dim(\mathrm{Col}(A)) = 6-2 = 4 < 5$

$\Rightarrow \mathrm{Col}(A) \neq \mathbb{R}^5 \Rightarrow$ It is possible there is no solution

Example :

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

(b) What about if we instead assume that the corresponding homogeneous linear system solution set is spanned by one non-zero vector? Justify your answer.

Solution:

If $\dim(\mathrm{Nul}(A)) = 1 \Rightarrow \dim(\mathrm{Col}(A)) = 6-1 = 5$

$\Rightarrow \mathrm{Col}(A) = \mathbb{R}^5 \Rightarrow A\underline{x} = \underline{b}$ always has a solution.

3. (25 points) Let $A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ and $B = \left\{ \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$. Find a basis C such that

$$A_{B,C} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}.$$

Solution:

$$A_{B,C} := \left((A\underline{b}_1)_C \ (A\underline{b}_2)_C \ (A\underline{b}_3)_C \ (A\underline{b}_4)_C \right)$$

$$= \left(\begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}_C \ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}_C \ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}_C \ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}_C \right)$$

$$A_{B,C} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \underline{c}_3 + 2\underline{c}_4$$

$$\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \underline{c}_3$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \underline{c}_2$$

$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = -\underline{c}_1$$

$$\Rightarrow \quad C = \left\{ \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1/2 \\ 0 \\ -1/2 \end{pmatrix} \right\}$$

4. (a) Give a precise statement of what it means for a square matrix A to be diagonalizable.
Solution:

A diagonalizable $\Leftrightarrow$ There exists P an invertible matrix such that $P^{-1}AP$ is diagonal

$\Leftrightarrow$ There exists a basis of eigenvectors of A

(b) Is the matrix $A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ diagonalizable? Justify your answer.

Solution:

A upper triangular $\Rightarrow$ $1, 3, 2$ are all eigenvalues of A

$\det(A - xI_4) = (1-x)(3-x)(2-x)^2 \Rightarrow$ algebraic multiplicity of 2 is 2

$$A - 2I_4 = \begin{pmatrix} \boxed{-1} & 2 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ $\dim(2\text{-eigenspace}) = 1 < 2 =$ algebraic multiplicity of 2

$\Rightarrow$ A is **not** diagonalizable.

5. Let W be the span of the vectors $\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$ in $\mathbb{R}^4$. Find an orthogonal basis for $W^\perp$.

What is the minimum distance between $\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$ and W?

Solution:

$$W = \text{Col}\left(\begin{pmatrix}1&0\\0&0\\1&1\\0&-1\end{pmatrix}\right) \Rightarrow W^\perp = \text{Nul}\begin{pmatrix}1&0&1&0\\0&0&1&-1\end{pmatrix}$$

$$\begin{pmatrix}1&0&1&0\\0&0&1&-1\end{pmatrix} \longrightarrow \begin{pmatrix}1&0&0&1\\0&0&1&-1\end{pmatrix} \Rightarrow \text{Nul}\begin{pmatrix}1&0&1&0\\0&0&1&-1\end{pmatrix} = \left\{\begin{pmatrix}-x_4\\x_2\\x_4\\x_4\end{pmatrix}\right\}$$

$$= \text{Span}\underbrace{\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\1\end{pmatrix}\right)}_{\text{Orthogonal basis for } W^\perp}$$

Min distance between $\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$ and W

$$= \left\| \text{Proj}_{W^\perp}\begin{pmatrix}1\\1\\1\\0\end{pmatrix} \right\|$$

$$= \left\| \frac{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\\0\end{pmatrix}}{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \frac{\begin{pmatrix}-1\\0\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\\0\end{pmatrix}}{\begin{pmatrix}-1\\0\\1\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\1\\1\end{pmatrix}}\begin{pmatrix}-1\\0\\1\\1\end{pmatrix} \right\|$$

$$= \left\| 1\cdot\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + 0\begin{pmatrix}-1\\0\\1\\1\end{pmatrix} \right\| = 1$$

END OF EXAM