MATH 54 MIDTERM 2 (PRACTICE 1)

PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.
CALCULATORS ARE NOT PERMITTED
YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK
SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE
REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT
THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name and Student ID: ______________________________

GSI's name: ______________________________

This exam consists of 5 questions. Answer the questions in the spaces provided.

1. (25 points) (a) Let V be a vector space and $U \subset V$ be a subset. Give a precise definition of what it means for U to be a subspace.
Solution:

$U \subset V$ is a subspace if

zero vector in V

1/ $\underline{0}$ in U

2/ If $\underline{x}, \underline{y}$ are in U then $\underline{x} + \underline{y}$ is in U

3/ If $\underline{x}$ in U and λ in $\mathbb{R}$ then $\lambda \underline{x}$ is in U

(b) If $V = C[0,1]$ is the vector space of continuous real-values functions on $[0,1]$ is it true that

$$U = \{f(x) \text{ in } C[0,1], \text{ such that } f(0) \text{ and } f(1) \text{ are integers}\}$$

is a subspace? Justify your answer.
Solution:

U is <u>not</u> a subspace. For example $f(x) = x$ is in U because $f(0) = 0$ and $f(1) = 1$. However $\frac{1}{2}f(x)$ is <u>not</u> in U as $f(1) = \frac{1}{2}$.

Not an integer

2. (25 points) Find a basis for the kernel of the following linear transformation:

$$\begin{aligned} T : \mathbb{P}_3(\mathbb{R}) &\to \mathbb{R}^3 \\ p(x) &\mapsto \begin{pmatrix} p(1) \\ p'(1) \\ p''(1) \end{pmatrix} \end{aligned}$$

Determine the rank of T. Be sure to carefully justify your answers.

Solution:

$\mathrm{Ker}(T) = \{ p(x) \text{ in } \mathbb{P}_3(\mathbb{R}) \text{ such that } T(p(x)) = \underline{0} \}$

$= \{ p(x) \text{ in } \mathbb{P}_3(\mathbb{R}) \text{ such that } p(1) = 0, p'(1) = 0, p''(1) = 0 \}$

Let $p(x) = a + bx + cx^2 + dx^3$

$p(1) = a + b + c + d$

$p'(1) = b + 2c + 3d$

$p''(1) = 2c + 6d$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 2 & 6 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right)$$

$\Rightarrow a = -d$, $b = 3d$, $c = -3d$ $\Rightarrow p(x) = -d + 3dx - 3dx^2 + dx^3$

$\Rightarrow \mathrm{Ker}(T) = \mathrm{Span}\big(\underbrace{-1 + 3x - 3x^2 + x^3}_{\text{Basis for Ker}(T)}\big)$

$\dim(\mathbb{P}_3(\mathbb{R})) = \dim(\mathrm{Ker}(T)) + \dim(\mathrm{Range}(T))$

$\Rightarrow 4 = 1 + \mathrm{Rank}(T) \Rightarrow \mathrm{Rank}(T) = 3$

3. Let V and W be vector spaces with bases $B = \{\underline{\mathbf{b}}_1, \underline{\mathbf{b}}_2, \underline{\mathbf{b}}_3\}$ and $C = \{\underline{\mathbf{c}}_1, \underline{\mathbf{c}}_2, \underline{\mathbf{c}}_3, \underline{\mathbf{c}}_4\}$ respectively. Let T be the linear transformation from V to W such that

$$T(\underline{\mathbf{b}}_1) = \underline{\mathbf{c}}_1 - 3\underline{\mathbf{c}}_4,\ T(\underline{\mathbf{b}}_2) = -\underline{\mathbf{c}}_3 + 2\underline{\mathbf{c}}_2,\ T(\underline{\mathbf{b}}_3) = \underline{\mathbf{c}}_1 + \underline{\mathbf{c}}_2 + \underline{\mathbf{c}}_3.$$

Determine the nullity of T. Is T one-to-one?

Solution:

$$A_{B,C} = \left((T(\underline{b}_1))_C \ \ (T(\underline{b}_2))_C \ \ (T(\underline{b}_3))_C \right)$$

$$= \left((\underline{c}_1 - 3\underline{c}_4)_C \ \ (-\underline{c}_3 + 2\underline{c}_2)_C \ \ (\underline{c}_1 + \underline{c}_2 + \underline{c}_3)_C \right)$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & 1 \\ -3 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & 1 \\ -3 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & \boxed{3} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{Nullity}(T) = \dim(\text{Ker}(T)) = \dim(\text{Nul}(A_{B,C})) = 0$$

No free columns

No free columns $\Rightarrow$ Columns of $A_{B,C}$ are L.I

$\Rightarrow$ T is one-to-one

4. Let $A = \begin{pmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$. Is it possible to find an invertible matrix P such that $P^{-1}AP$ is diagonal. If so, find such a P. If not, justify why.

Solution:

$$\det(A - xI_4) = (5-x)(3-x)(2-x)^2 \Rightarrow 5, 3, 2 \text{ are all eigenvalues}$$

algebraic multiplicity of 5 $= 1 \Rightarrow \dim(5\text{-eigenspace}) = 1$

$$A - 5I_4 = \begin{pmatrix} 0 & -3 & 0 & 9 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ basis for } 5\text{-eigenspace}$$

algebraic multiplicity of 3 $= 1 \Rightarrow \dim(3\text{-eigenspace}) = 1$

$$A - 3I_4 = \begin{pmatrix} 2 & -3 & 0 & 9 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 2 & -3 & 0 & 9 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} \text{ basis for } 3\text{-eigenspace}$$

$$A - 2I_4 = \begin{pmatrix} 3 & -3 & 0 & 9 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow 2\text{-eigenspace} = \left\{ \begin{pmatrix} -x_3 - x_4 \\ -x_3 + 2x_4 \\ x_3 \\ x_4 \end{pmatrix} \right\} = \mathrm{Span}\Bigg(\underbrace{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}}_{\text{Basis}} \Bigg)$$

$\Rightarrow$ A diagonalizable and if $P = \begin{pmatrix} 1 & 3 & -1 & -1 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ then

$$P^{-1}AP = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

5. Let W be the span of the vectors $\underset{\shortparallel\atop \underline{x}_1}{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}}, \underset{\shortparallel\atop \underline{x}_2}{\begin{pmatrix}0\\1\\1\\0\end{pmatrix}}$ in $\mathbb{R}^4$. Calculate $Proj_W\begin{pmatrix}1\\-1\\2\\1\end{pmatrix}$ and $Proj_{W^\perp}\begin{pmatrix}1\\-1\\2\\1\end{pmatrix}$?.

Solution:

$$\underline{v}_1 = \underline{x}_1$$

$$\underline{v}_2 = \underline{x}_2 - \frac{\underline{v}_1 \cdot \underline{x}_2}{\underline{v}_1 \cdot \underline{v}_1}\underline{v}_1 = \begin{pmatrix}0\\1\\1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}-1/2\\1/2\\1\\0\end{pmatrix}$$

$$\Rightarrow \quad W = \mathrm{Span}\left(\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\1\\2\\0\end{pmatrix}\right)$$

$$\Rightarrow \quad Proj_W\begin{pmatrix}1\\-1\\2\\1\end{pmatrix} = \frac{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\2\\1\end{pmatrix}}{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\\0\end{pmatrix}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \frac{\begin{pmatrix}-1\\1\\2\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\2\\0\end{pmatrix}}{\begin{pmatrix}-1\\1\\2\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\2\\0\end{pmatrix}}\cdot\begin{pmatrix}-1\\1\\2\\0\end{pmatrix}$$

$$= \frac{2}{6}\cdot\begin{pmatrix}-1\\1\\2\\0\end{pmatrix} = \begin{pmatrix}-1/3\\1/3\\2/3\\0\end{pmatrix}$$

$$Proj_{W^\perp}\begin{pmatrix}1\\-1\\2\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\2\\1\end{pmatrix} - \begin{pmatrix}-1/3\\1/3\\2/3\\0\end{pmatrix} = \begin{pmatrix}4/3\\-4/3\\4/3\\1\end{pmatrix}$$

END OF EXAM