MATH 54 MIDTERM 2 (002)

PROFESSOR PAULIN

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CALCULATORS ARE NOT PERMITTED
YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK
SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE
REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT
THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name and Student ID: ______________________________

GSI's name: ______________________________

This exam consists of 5 questions. Answer the questions in the spaces provided.

1. (25 points) (a) Let $A = \begin{pmatrix} 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ -3 & -3 & 6 & 9 & 0 \\ 0 & 0 & 3 & 3 & -1 \end{pmatrix}$. Calculate bases for $Ker(T_A)$ and $Range(T_A)$.

Solution:

$$\begin{pmatrix} 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ -3 & -3 & 6 & 9 & 0 \\ 0 & 0 & 3 & 3 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 6 & 6 & 3 \\ 0 & 0 & 3 & 3 & -1 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & 1 & 0 & -1 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

basis

$$\Rightarrow \quad Range(T_A) = Span\left(\begin{pmatrix} 1 \\ 0 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right)$$

basis

$$Ker(T_A) = \left\{ \begin{pmatrix} -x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \\ 0 \end{pmatrix} \right\} = Span\left(\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right)$$

(b) What is $Rank(T_A)$? What is $Nullity(T_A)$?

Solution:

$$Rank(T_A) = \dim(Range(T_A)) = 3$$

$$Nullity(T_A) = \dim(Ker(T_A)) = 2$$

2. (25 points) Let

$$B = \{1, 1+x, 1+x+x^2\} \subset \mathbb{P}_2(\mathbb{R}), \quad C = \{1, 1+x, x+x^2, x^2+x^3\} \subset \mathbb{P}_3(\mathbb{R})$$

be bases respectively. Let T be the following linear transformation:

$$\begin{aligned} T : \mathbb{P}_2(\mathbb{R}) &\to \mathbb{P}_3(\mathbb{R}) \\ p(x) &\mapsto xp(x) + p'(x) \end{aligned}$$

Determine the matrix of T with respect to bases B and C. Is T one-to-one? Justify your answer.

Solution:

$T(1) = x = (-1)\cdot 1 + 1\cdot(1+x) + 0(x+x^2) + 0(x^2+x^3)$

$T(1+x) = x(1+x) + 1 = 1 + x + x^2$
$= 1\cdot 1 + 0\cdot(1+x) + 1\cdot(x+x^2) + 0(x^2+x^3)$

$T(1+x+x^2) = x(1+x+x^2) + (1+2x) = 1+3x+x^2+x^3$
$= (-2)\cdot 1 + 3(1+x) + 0(x+x^2) + 1(x^2+x^3)$

$\Rightarrow \quad A_{B,C} = \left((T(\underline{b_1}))_C \; T(\underline{b_2})_C \; T(\underline{b_3})_C \right) = \begin{pmatrix} -1 & 1 & -2 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} -1 & 1 & -2 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$ Pivot in every column $\Rightarrow T$ one-to-one

3. (25 points) Let $A = \begin{pmatrix} 1 & 2 & x & 2 \\ 0 & 2 & 1 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. For what values of x and y is A diagonalizable? You do not need to diagonalize A. Carefully justify your answer. Hint: Consider algebraic multiplicities.

Solution:

A upper triangular $\Rightarrow$ 1, 2 are eigenvalues

algebraic multiplicity of $2 = 1 = \dim(2\text{-eigenspace})$

algebraic multiplicity of $1 = 3$. Need x, y such that $\dim(1\text{-eigenspace}) = 3$

$$A - 1 \cdot I_4 = \begin{pmatrix} 0 & 2 & x & 2 \\ 0 & 1 & 1 & y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 2 & x & 2 \\ 0 & 0 & 1-\frac{1}{2}x & y-1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\dim(1\text{-eigenspace}) = 3 \iff$ 3 free columns $\iff 1 - \frac{1}{2}x = 0$ and $y - 1 = 0$

$\iff x = 2,\ y = 1$

$\Rightarrow$ A diagonalizable $\iff x = 2,\ y = 1$

4. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & -1 & -1 \\ 2 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ and $C = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\}$. Find a basis B such that

$$A_{B,C} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solution:

$$A_{B,C} = \left((A\underline{b}_1)_C \ (A\underline{b}_2)_C \ (A\underline{b}_3)_C \right) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \quad A\underline{b}_1 = \underline{0}, \quad A\underline{b}_2 = \underline{c}_1 - \underline{c}_2, \quad A\underline{b}_3 = \underline{c}_1 + \underline{c}_3$

← in Nul(A) ← 3rd column of A ← 2nd column of A

$\Rightarrow \quad A\underline{b}_1 = \underline{0}, \quad A\underline{b}_2 = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \quad A\underline{b}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & -1 & -1 \\ 2 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -3 & -1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \operatorname{Nul}(A) = \operatorname{Span}\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

Choose $\underline{b}_1 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \quad \underline{b}_2 = \underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \underline{b}_3 = \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

5. (25 points) Let W be the span of the vectors $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ in $\mathbb{R}^5$.

Calculate the minimum distance between $\begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and W?. Hint: This can be done without using the Gram-Schmidt process.

Solution:

$$W^{\perp} = \text{Nul} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \Rightarrow W^{\perp} = \text{Span} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Min distance between $\begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and W

$$= \left\| \text{Proj}_{W^{\perp}} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\|$$

$$= \left\| \frac{5}{5} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\| = \sqrt{5}$$

END OF EXAM