# MATH 54 MIDTERM 2 (001)

## PROFESSOR PAULIN

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| **DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.** |
| **CALCULATORS ARE NOT PERMITTED** |
| **YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK** |
| **SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE** |
| **REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT** |
| **THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM** |

Name and Student ID: ______________________________

GSI's name: ______________________________

**This exam consists of 5 questions. Answer the questions in the spaces provided.**

1. (25 points) (a) Let $A = \begin{pmatrix} 0 & 1 & -1 & 1 & 1 & 3 \\ 0 & -1 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 2 & -2 & 6 \\ 0 & 3 & -3 & 3 & 3 & 8 \end{pmatrix}$. Calculate bases for $Ker(T_A)$ and $Range(T_A)$.

**Solution:**

$$\begin{pmatrix} 0 & 1 & -1 & 1 & 1 & 3 \\ 0 & -1 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 2 & -2 & 6 \\ 0 & 3 & -3 & 3 & 3 & 8 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & -1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 & -2 & 6 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 0 & \boxed{1} & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Basis

$$\Rightarrow \quad Range(T_A) = Span\left( \begin{pmatrix} 1 \\ -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 6 \\ 8 \end{pmatrix} \right)$$

Basis

$$Ker(T_A) = \left\{ \begin{pmatrix} x_1 \\ x_3 - 2x_5 \\ x_3 \\ x_5 \\ x_5 \\ 0 \end{pmatrix} \right\} = Span\left( \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right)$$

(b) What is $Rank(T_A)$? What is $Nullity(T_A)$?

**Solution:**

$Rank(T_A) = \dim(Range(T_A)) = 3$

$Nullity(T_A) = \dim(Ker(T_A)) = 3$

2. (25 points) Let

$$B = \{1, 1-x, 1+x-x^2, x^3-x\} \subset \mathbb{P}_3(\mathbb{R}), \quad C = \{x^2-x, x+2, -2\} \subset \mathbb{P}_2(\mathbb{R})$$

be bases respectively. Let $T$ be the following linear transformation:

$$\begin{aligned} T : \mathbb{P}_3(\mathbb{R}) &\to \mathbb{P}_2(\mathbb{R}) \\ p(x) &\mapsto xp''(x) - p'(x) \end{aligned}$$

Determine the matrix of $T$ with respect to bases $B$ and $C$. Is $T$ onto? Justify your answer.

**Solution:**

$$T(1) = 0 = 0(x^2-x) + 0(x+2) + 0(-2)$$

$$T(1-x) = 1 = 0(x^2-x) + 0(x+2) + (-\tfrac{1}{2})(-2)$$

$$T(1+x-x^2) = x(-2) - (1-2x) = -1 = 0(x^2-x) + 0(x+2) + \tfrac{1}{2}(-2)$$

$$T(x^3-x) = x(6x) - (3x^2-1) = 3x^2+1 = 3(x^2-x) + 3(x+2) + (\tfrac{5}{2})(-2)$$

$$\Rightarrow \quad A_{B,C} = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} \end{pmatrix} \to \begin{pmatrix} 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ Not a pivot in every row

$\Rightarrow$ T not onto

3. Let $A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & a & 1 \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 2 \end{pmatrix}$. For what values of $a$ and $b$ is $A$ diagonalizable? You do not need to diagonalize $A$. Justify your answer.

**Solution:**

1, 2 = eigenvalues of $A$ Algebraic multiplicity of 1, 2 = 2

$\Rightarrow$ $A$ diagonalizable $\Leftrightarrow$ $\dim(\mathrm{Nul}(A - 1 \cdot I_4)) = \dim(\mathrm{Nul}(A - 2I_4)) = 2$

$$A - 1 \cdot I_4 = \begin{pmatrix} 0 & -1 & 2 & 1 \\ 0 & 1 & a & 1 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -1 & 2 & 1 \\ 0 & 0 & a+2 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ $\dim(\mathrm{Nul}(A - 1 \cdot I_4)) = 2 \Leftrightarrow a = -2$ (2 free columns)

Let $a = -2$

$$A - 2 \cdot I_4 = \begin{pmatrix} -1 & -1 & 2 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -1 & b \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & -1 & 2 & 1 \\ 0 & 0 & -1 & b \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & \end{pmatrix} \rightarrow \begin{pmatrix} -1 & -1 & 2 & 1 \\ 0 & 0 & -1 & b \\ 0 & 0 & 0 & 1-2b \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ $\dim(\mathrm{Nul}(A - 2 \cdot I_4)) = 2 \Leftrightarrow 1 - 2b = 0 \Leftrightarrow b = \frac{1}{2}$ (2 free columns)

$\Rightarrow$ $A$ diagonalizable $\Leftrightarrow$ $a = -2$, $b = 1/2$

4. Let $A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ and $C = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$. Find a basis $B$ such that

$$A_{B,C} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

**Solution:**

$$A_{B,C} = \left( (A\underline{b}_1)_C \ (A\underline{b}_2)_C \ (A\underline{b}_3)_C \right) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Leftrightarrow \quad (A\underline{b}_1)_C = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad (A\underline{b}_2)_C = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad (A\underline{b}_3)_C = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \quad A\underline{b}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad A\underline{b}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad A\underline{b}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

2nd column of $A$

3rd column of $A$

$\underline{b}_3$ in Nul$(A)$

$$\begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{Nul}(A) = \text{Span}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \text{Choose} \quad \underline{b}_1 = \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \underline{b}_2 = \underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ \underline{b}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

5. Let $W$ be the span of the vectors $\begin{pmatrix}1\\2\\0\\-3\end{pmatrix}$, $\begin{pmatrix}0\\-2\\1\\1\end{pmatrix}$, $\begin{pmatrix}0\\1\\3\\-4\end{pmatrix}$ in $\mathbb{R}^4$. Calculate the minimum distance between $\begin{pmatrix}3\\0\\1\\0\end{pmatrix}$ and $W$? Hint: This problem can be solved without applying Gram-Schmidt.

**Solution:**

$$W^{\perp} = \mathrm{Nul}\left(\begin{pmatrix}1&2&0&-3\\0&-2&1&1\\0&1&3&-4\end{pmatrix}\right)$$

$$\begin{pmatrix}1&2&0&-3\\0&-2&1&1\\0&1&3&-4\end{pmatrix} \longrightarrow \begin{pmatrix}1&2&0&-3\\0&1&3&-4\\0&-2&1&1\end{pmatrix} \longrightarrow \begin{pmatrix}1&2&0&-3\\0&1&3&-4\\0&0&7&-7\end{pmatrix}$$

$$\downarrow$$

$$W^{\perp} = \mathrm{Span}\left(\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right) \Leftarrow \begin{pmatrix}1&0&0&-1\\0&1&0&-1\\0&0&1&-1\end{pmatrix}$$

Min distance between $W$ and $\begin{pmatrix}3\\0\\1\\0\end{pmatrix}$ $= \left\| \mathrm{Proj}_{W^{\perp}}\begin{pmatrix}3\\0\\1\\0\end{pmatrix}\right\| = \left\| \frac{4}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\|$

$= \sqrt{4} = 2$

END OF EXAM