MATH 54 MIDTERM 1 (PRACTICE 3)

PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.
CALCULATORS ARE NOT PERMITTED
YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK
SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE
REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT
THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name and section: ____________________

GSI's name: ____________________

This exam consists of 5 questions. Answer the questions in the spaces provided.

1. (25 points) (a) Express the vector $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ as a linear combination of the vectors

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Solution:

$$\left(\begin{array}{cccc|c} 0 & 2 & 2 & 0 & -2 \\ 1 & 3 & 1 & 1 & 0 \\ 1 & 4 & 2 & 2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 & -2 \\ 1 & 4 & 2 & 2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 & -2 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 0 & -2 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \longleftarrow \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \longleftarrow \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \Rightarrow \begin{aligned} x_1 - 2x_3 &= 1 \\ x_2 + x_3 &= -1 \\ x_4 &= 2 \end{aligned} \Rightarrow \begin{aligned} x_1 &= 1 + 2x_3 \\ x_2 &= -1 - x_3 \\ x_3 &\text{ free} \\ x_4 &= 2 \end{aligned}$$

$$x_3 = 0 \Rightarrow \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

(b) How many possible ways of doing this are there? Justify your answer.

Solution:

There are infinitely many ways to do this as x_3 is a free variable.

2. (25 points) List the possible forms of all 3×3 reduced echelon matrices. Label the ones row equivalent to matrices whose column vectors are linearly dependent. Label the ones row equivalent to matrices whose column vectors span $\mathbb{R}^3$.

Solution:

$$\begin{pmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{pmatrix}$$

Span $= \mathbb{R}^3$

$$\begin{pmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & 0 \end{pmatrix}$$

L. D.

$$\begin{pmatrix} \blacksquare & * & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{pmatrix}$$

L. D.

$$\begin{pmatrix} \blacksquare & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

L. D.

$$\begin{pmatrix} 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{pmatrix}$$

L. D.

$$\begin{pmatrix} 0 & \blacksquare & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

L. D.

$$\begin{pmatrix} 0 & 0 & \blacksquare \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

L. D.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

L. D.

3. Calculate the determinant of $\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 2 & 0 \\ 2 & 1 & 3 & 4 \\ 3 & 1 & 2 & 5 \end{pmatrix}$.

Solution:

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 2 & 0 \\ 2 & 1 & 3 & 4 \\ 3 & 1 & 2 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 2 & -1 \\ 0 & -1 & 3 & 2 \\ 0 & -2 & 2 & 2 \end{pmatrix} \xrightarrow{\text{Switch } 2^{nd} \text{ and } 3^{rd}} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & -2 & 2 & -1 \\ 0 & -2 & 2 & 2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & -4 & -5 \\ 0 & 0 & -4 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & -4 & -5 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

one row switch

$$\Rightarrow \begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 2 & 0 \\ 2 & 0 & 3 & 4 \\ 3 & 1 & 2 & 5 \end{vmatrix} = (-1) \cdot 1 \cdot -1 \cdot -4 \cdot 3 = -12$$

4. (25 points) (a) Is it possible for a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ to be one-to one? Justify your answer.

Solution:

It is not possible for $T : \mathbb{R}^4 \to \mathbb{R}^3$ linear to be one-to-one.

T linear $\Rightarrow$ $T = T_A$ for some A, a 3×4 matrix.

T_A one-to-one $\Leftrightarrow$ Reduced A has pivot position in every column

However A 3×4 $\Rightarrow$ At most 3 pivot positions.

There are 4 columns so there cannot be a pivot in every column.

(b) Give an example of a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ which is onto.

Solution:

$$T = T_A \quad \text{where} \quad A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

In reduced form with pivot position in every row $\Rightarrow$ T_A onto.

5. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation given by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_1 + 3x_3 + x_4 \\ -x_1 + x_2 \\ x_3 + x_4 \\ x_1 - x_4 \end{pmatrix}.$$

Give an example of $\underline{\mathbf{b}}$ in $\mathbb{R}^4$ not in the range of T? Justify your answer.

Solution:

$$T(\underline{x}) = \underbrace{\begin{pmatrix} 2 & 0 & 3 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}}_{A} \underline{x}$$

$\underline{b}$ not in range of $T \iff (A \mid \underline{b})$ inconsistent

$$\left(\begin{array}{cccc|c} 2 & 0 & 3 & 1 & b_1 \\ -1 & 1 & 0 & 0 & b_2 \\ 0 & 0 & 1 & 1 & b_3 \\ 1 & 0 & 0 & -1 & b_4 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & b_4 \\ -1 & 1 & 0 & 0 & b_2 \\ 0 & 0 & 1 & 1 & b_3 \\ 2 & 0 & 3 & 1 & b_1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & b_4 \\ 0 & 1 & 0 & -1 & b_2+b_4 \\ 0 & 0 & 1 & 1 & b_3 \\ 0 & 0 & 0 & 0 & b_1-2b_4-3b_3 \end{array}\right) \longleftarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & b_4 \\ 0 & 1 & 0 & -1 & b_2+b_4 \\ 0 & 0 & 1 & 1 & b_3 \\ 0 & 0 & 3 & 3 & b_1-2b_4 \end{array}\right)$$

Let $b_1 = 1$, $b_2 = 1$, $b_3 = 0$, $b_4 = 0 \Rightarrow b_1 - 2b_4 - 3b_3 \neq 0$

$\Rightarrow (A \mid \underline{b})$ inconsistent $\Rightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ not in range of T

END OF EXAM