MATH 54 MIDTERM 1 (PRACTICE 2)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name and section: ______________________________

GSI's name: ______________________________

This exam consists of 5 questions. Answer the questions in the spaces provided.

1. (25 points) (a) Determine the value of k such $\begin{pmatrix} 2 \\ 3 \\ k \end{pmatrix}$ can be written as a linear combination of the vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$. For this value express is as a linear combination.

Solution:

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 1 & 1 & 5 & 3 \\ 0 & 1 & 3 & k \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & k \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1-k \end{array}\right)$$

$\Rightarrow \begin{pmatrix} 2 \\ 3 \\ k \end{pmatrix}$ can be expressed a linear combination $\Longleftrightarrow k = 1$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 + 2x_3 = 2 \\ x_2 + 3x_3 = 1 \\ x_3 \text{ free} \end{array}$$

$\Rightarrow \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ is a solution. Hence

$$2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

(b) Give an example of a vector $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, such that $\left(\begin{array}{ccc|c} 1 & 0 & 2 & b_1 \\ 1 & 1 & 5 & b_2 \\ 0 & 1 & 3 & b_3 \end{array}\right)$ is inconsistent.

Solution:

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

2. (25 points) Show that

$$\mathrm{Span}\left(\begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}3\\7\\14\end{pmatrix}, \begin{pmatrix}1\\3\\7\end{pmatrix}\right) = \mathbb{R}^3.$$

Are these vectors linearly independent? Justify your answers.

Solution:

$$\begin{pmatrix}1 & 0 & 3 & 1\\ 1 & 1 & 7 & 3\\ 2 & 2 & 14 & 7\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 0 & 3 & 1\\ 0 & 1 & 4 & 2\\ 0 & 2 & 8 & 5\end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix}\boxed{1} & 0 & 3 & 1\\ 0 & \boxed{1} & 4 & 2\\ 0 & 0 & 0 & \boxed{1}\end{pmatrix}$$

$\Rightarrow$ Pivot position in <u>every row</u> $\Rightarrow$ $\mathrm{Span}\left(\begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}3\\7\\14\end{pmatrix}, \begin{pmatrix}1\\3\\7\end{pmatrix}\right) = \mathbb{R}^3$

There is not a pivot position in <u>every column</u> $\Rightarrow$ $\left\{\begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}3\\7\\14\end{pmatrix}, \begin{pmatrix}1\\3\\7\end{pmatrix}\right\}$

Linearly dependent

3. Let
$$A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 2 & 3 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Calculate $det(AB)$ and $det(BA)$.

Solution:

$$AB = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 5 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 5 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & -9 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \det(AB) = 0$$

$$BA = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 2 & -1 \end{pmatrix}$$

$$\Rightarrow \det(BA) = 1 \cdot (-1) - 5 \cdot 2 = -11$$

4. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ has the property that

$$T\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\2\\4\end{pmatrix}, \quad T\begin{pmatrix}0\\-3\\0\end{pmatrix} = \begin{pmatrix}-3\\0\\6\end{pmatrix}, \quad T\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\5\\-2\end{pmatrix}.$$

Calculate the standard matrix associated to T. Is T one-to-one? Justify your answer.

Solution:

$$T\begin{pmatrix}0\\-3\\0\end{pmatrix} = \begin{pmatrix}-3\\0\\6\end{pmatrix} \Rightarrow -3T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-3\\0\\6\end{pmatrix}$$

$$\Rightarrow T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\-2\end{pmatrix}$$

linear

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = T\left(\begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix}\right) = T\begin{pmatrix}1\\1\\0\end{pmatrix} - T\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}1\\2\\4\end{pmatrix} - \begin{pmatrix}1\\0\\-2\end{pmatrix} = \begin{pmatrix}0\\2\\6\end{pmatrix}$$

$$T\begin{pmatrix}0\\0\\1\end{pmatrix} = T\left(\begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix}\right) = T\begin{pmatrix}0\\1\\1\end{pmatrix} - T\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\5\\-2\end{pmatrix} - \begin{pmatrix}1\\0\\-2\end{pmatrix} = \begin{pmatrix}-1\\5\\0\end{pmatrix}$$

$\Rightarrow$ Standard Matrix $= \begin{pmatrix}0 & 1 & -1\\2 & 0 & 5\\6 & -2 & 0\end{pmatrix}$

$$\begin{pmatrix}0 & 1 & -1\\2 & 0 & 5\\6 & -2 & 0\end{pmatrix} \longrightarrow \begin{pmatrix}2 & 0 & 5\\0 & 1 & -1\\6 & -2 & 0\end{pmatrix} \longrightarrow \begin{pmatrix}2 & 0 & 5\\0 & 1 & -1\\0 & -2 & -15\end{pmatrix}$$

$$\downarrow$$

T <u>one-to-one</u> $\Leftarrow$ Pivot Position in <u>every column</u> $\Leftarrow \begin{pmatrix}\boxed{2} & 0 & 5\\0 & \boxed{1} & -1\\0 & 0 & \boxed{-17}\end{pmatrix}$

5. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation given by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + x_3 - x_4 \\ x_2 + 2x_4 \\ x_1 + 2x_4 \\ x_1 + x_3 - 2x_4 \end{pmatrix}.$$

Find a linear transformation $S : \mathbb{R}^4 \to \mathbb{R}^4$ such that $T \circ S = S \circ T = Id_{\mathbb{R}^4}$.

Solution:

$$T = T_A \quad \text{where} \quad A = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & -2 \end{pmatrix}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & -2 & 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 2 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 4 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \end{array}\right) \longleftarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 4 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \end{array}\right) \Rightarrow S\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_1 + x_3 + 2x_4 \\ -2x_1 + x_2 + 2x_4 \\ 4x_1 - x_3 - 3x_4 \\ x_1 - x_4 \end{pmatrix}$$

has desired properties.

END OF EXAM