MATH 54 MIDTERM 1 (PRACTICE 1)

PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.
CALCULATORS ARE NOT PERMITTED
YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK
SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE
REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT
THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name and section: ______________________________

GSI's name: ______________________________

This exam consists of 5 questions. Answer the questions in the spaces provided.

1. (25 points) (a) Calculate the general solution to the linear system with the following augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 4 & 0 & 3 \\ 3 & 12 & 3 & 15 \\ 2 & 8 & 1 & 8 \end{array}\right)$$

Solution:

$$\left(\begin{array}{ccc|c} 1 & 4 & 0 & 3 \\ 3 & 12 & 3 & 15 \\ 2 & 8 & 1 & 8 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 4 & 0 & 3 \\ 1 & 4 & 1 & 5 \\ 2 & 8 & 1 & 8 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 4 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\downarrow$$

$$\begin{array}{l} x_1 + 4x_2 = 3 \\ x_3 = 2 \end{array} \Leftarrow \begin{array}{c} \left(\begin{array}{ccc|c} 1 & 4 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{ccc} P & F & P \end{array} \end{array}$$

Hence general solution is $\begin{pmatrix} 3-4x_2 \\ x_2 \\ 2 \end{pmatrix}$ where x_2 is free.

(b) Write down a general solution to the associated homogeneous linear system.
Solution:

$$\begin{pmatrix} 3-4x_2 \\ x_2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3-4x_2 \\ 0+x_2 \\ 2+0\cdot x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $x_2 \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$ is a general solution to homogeneous problem.

2. (25 points) Determine all values of h such that the vectors $\begin{pmatrix}1\\2\\0\\2\end{pmatrix}, \begin{pmatrix}0\\h+1\\0\\h+1\end{pmatrix}, \begin{pmatrix}3\\10\\1\\10\end{pmatrix}, \begin{pmatrix}1\\3\\h\\h+3\end{pmatrix}$ are linearly dependent.

Solution:

$$\begin{pmatrix}1 & 0 & 3 & 1\\ 2 & h+1 & 10 & 3\\ 0 & 0 & 1 & h\\ 2 & h+1 & 10 & h+3\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 0 & 3 & 1\\ 0 & h+1 & 4 & 1\\ 0 & 0 & 1 & h\\ 0 & h+1 & 4 & h+1\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 0 & 3 & 1\\ 0 & h+1 & 4 & 1\\ 0 & 0 & 1 & h\\ 0 & 0 & 0 & h\end{pmatrix}$$

↑ Echelon Form

Vectors independent $\iff$ Every column contains a pivot position

$\Updownarrow$

$h+1 \neq 0$ and $h \neq 0$

$\Updownarrow$

$h \neq -1$ and $h \neq 0$

Hence vectors are linearly dependent iff $h=-1$ or $h=0$

3. Give an example of two linear transformations S and T from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $S \circ T \neq T \circ S$. Hint: Consider 2×2 matrices.

Solution:

Recall that given A, B, 2×2 matrices

$$\underbrace{T_A \circ T_B}_{\text{Both linear}} = T_{AB} \quad \text{and} \quad T_B \circ T_A = T_{BA} \quad \underline{\text{and}} \quad T_{AB} = T_{BA} \Leftrightarrow AB = BA$$

Thus we seek A, B such that $AB \neq BA$.

Almost any random choice will do.

Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, $B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$$

$$\neq$$

$$BA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$$

Let $T = T_A$, $S = T_B$ $\Rightarrow T \circ S \neq S \circ T$

4. If it exists, calculate the inverse matrix of $\begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 4 & 5 \\ 1 & 1 & 3 & 7 \\ 1 & 0 & 1 & -2 \end{pmatrix}$.

Solution:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 4 & 5 & 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 7 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & -2 & 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 6 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 8 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 6 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \end{array}\right) \longleftarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 6 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 2 & 0 & 0 & -1 \\ 0 & 1 & 3 & 0 & -7 & 1 & 0 & 6 \\ 0 & 0 & 1 & 0 & 2 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -13 & -2 & 3 & 12 \\ 0 & 0 & 1 & 0 & 2 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 4 & 5 \\ 1 & 1 & 3 & 7 \\ 1 & 0 & 1 & -2 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 & 1 & 1 \\ -13 & -2 & 3 & 12 \\ 2 & 1 & -1 & -2 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Is the set

$$T\begin{pmatrix}1\\2\\-1\end{pmatrix}, T\begin{pmatrix}3\\5\\0\end{pmatrix}, T\begin{pmatrix}1\\1\\2\end{pmatrix}$$

linearly independent? Justify your answer. Hint: $T(\underline{\mathbf{0}}) = \underline{\mathbf{0}}$.

Solution:

$$\begin{pmatrix}1 & 3 & 1\\ 2 & 5 & 1\\ -1 & 0 & 2\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 3 & 1\\ 0 & -1 & -1\\ 0 & 3 & 3\end{pmatrix} \longrightarrow \begin{pmatrix}\boxed{1} & 3 & 1\\ 0 & \boxed{1} & 1\\ 0 & 0 & 0\end{pmatrix}$$

$\Rightarrow$ There is <u>not</u> a pivot position in every column

$\Rightarrow$ $\begin{pmatrix}1\\2\\-1\end{pmatrix}, \begin{pmatrix}3\\5\\0\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}$ are linearly dependent.

$\Rightarrow$ There exist $\lambda_1, \lambda_2, \lambda_3$ <u>not all zero</u> such that

$$\lambda_1\begin{pmatrix}1\\2\\-1\end{pmatrix} + \lambda_2\begin{pmatrix}3\\5\\0\end{pmatrix} + \lambda_3\begin{pmatrix}1\\1\\2\end{pmatrix} = \underline{0}$$

$$\Rightarrow T\left(\lambda_1\begin{pmatrix}1\\2\\-1\end{pmatrix} + \lambda_2\begin{pmatrix}3\\5\\0\end{pmatrix} + \lambda_3\begin{pmatrix}1\\1\\2\end{pmatrix}\right) = T(\underline{0}) = \underline{0}$$

$$\Rightarrow \lambda_1 T\begin{pmatrix}1\\2\\-1\end{pmatrix} + \lambda_2 T\begin{pmatrix}3\\5\\0\end{pmatrix} + \lambda_3 T\begin{pmatrix}1\\1\\2\end{pmatrix} = \underline{0}$$

$\Rightarrow$ $T\begin{pmatrix}1\\2\\-1\end{pmatrix}, T\begin{pmatrix}3\\5\\0\end{pmatrix}, T\begin{pmatrix}1\\1\\2\end{pmatrix}$ are <u>linearly dependent</u>.

END OF EXAM