MATH 54 MIDTERM 1 (002)

PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.
CALCULATORS ARE NOT PERMITTED
YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK
SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE
REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT
THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name and section: ______________________________

GSI's name: ______________________________

This exam consists of 5 questions. Answer the questions in the spaces provided.

1. (25 points) (a) Calculate the general solution to the linear system with the following augmented matrix:

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 0 & -1 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 1 & -1 & -1 & -1 \end{array}\right)$$

Solution:

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 0 & -1 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 1 & -1 & -1 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right) \longrightarrow \begin{array}{c} \begin{array}{cccccc} & & & F & F & \\ & & & \downarrow & \downarrow & \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -2 & -1 & -2 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right) \end{array}$$

$$\Rightarrow \quad \text{General Solution} = \begin{pmatrix} -2 + 2x_4 + x_5 \\ 1 - x_4 - x_5 \\ 1 - x_4 + x_5 \\ x_4 \\ x_5 \end{pmatrix}$$

Free

(b) Is it possible for a linear system with the above coefficient matrix to have a unique solution? Justify your answer.

Solution:

It is not possible as x_4 and x_5 are always free variables, so if a system is consistent it must have infinitely many solutions.

2. (25 points) . Let $A = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 1 & 2 \\ -1 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & 0 \\ 3 & -4 & 0 \\ -1 & 1 & 1 \end{pmatrix}$. Calculate AB. Using this, or otherwise, calculate the determinant and inverse of AB.

Solution:

$$\begin{pmatrix} 1 & -1 & 0 \\ -2 & 1 & 2 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 3 & -4 & 0 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 3 & 0 \\ -1 & 0 & 2 \\ 1 & -2 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} -2 & 3 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{Switch } 1^{st} \text{ and } 3^{rd}} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 0 & 1 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ -2 & 3 & 0 & 1 & 0 & 0 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 0 & 1 \\ 0 & -1 & 2 & 1 & 0 & 2 \\ 0 & -2 & 3 & 0 & 1 & 1 \end{array}\right) \xleftarrow{\text{Switch } 2^{nd}, 3^{rd}} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 0 & 1 \\ 0 & -2 & 3 & 0 & 1 & 1 \\ 0 & -1 & 2 & 1 & 0 & 2 \end{array}\right)$$

$$\downarrow$$

$\Rightarrow$ determinant $= (-1)^2 \cdot 1 \cdot (-1) \cdot (-1) = 1$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 0 & 1 \\ 0 & -1 & 2 & 1 & 0 & 2 \\ 0 & 0 & -1 & -2 & 1 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & -2 & 1 & -2 \\ 0 & -1 & 0 & -3 & 2 & -4 \\ 0 & 0 & -1 & -2 & 1 & -3 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -3 & 6 \\ 0 & 1 & 0 & 3 & -2 & 4 \\ 0 & 0 & 1 & 2 & -1 & 3 \end{array}\right) \longleftarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -3 & 6 \\ 0 & -1 & 0 & -3 & 2 & -4 \\ 0 & 0 & -1 & -2 & 1 & -3 \end{array}\right)$$

Inverse

3. (25 points) Find all possible value of a, b, c such that $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is a solution to linear system

$$\left(\begin{array}{ccc|c} a & 2b & -2 & 0 \\ b & -c & a & 2 \end{array}\right)$$

Solution:

$$2\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 2b \\ -c \end{pmatrix} + \begin{pmatrix} -2 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \Rightarrow \begin{array}{l} 2a + 2b = 2 \\ a + 2b - c = 2 \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & 2 & 0 & 2 \\ 1 & 2 & -1 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 2 & -1 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$\Rightarrow$ $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ solution $\Leftrightarrow$ $\begin{array}{l} a = -c \\ b = 1 + c \\ c \text{ free} \end{array}$

4. (25 points) (a) Let A be a 4×4 matrix with the following properties:
The first column is non-zero. The first three columns are linearly dependent.
Write the echelon form matrices which are potentially row equivalent to A.
Solution:

$$\begin{pmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) Let A and B be two matrices satisfying the above properties. If, in both cases, the last column is a linear combination of the first three columns, must A and B be row equivalent to each other? Justify your answer.
Solution:

No : $A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Are different reduced echelon matrices so cannot be row equivalent.

5. (25 points) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation such that

$$T\begin{pmatrix}0\\-1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\-t\\-1\\t\end{pmatrix},\ T\begin{pmatrix}1\\-1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\-t\\0\\t\end{pmatrix},\ T\begin{pmatrix}0\\0\\-1\\0\end{pmatrix} = \begin{pmatrix}-2\\-3\\-t-1\\3\end{pmatrix},\ T\begin{pmatrix}0\\1\\1\\1\end{pmatrix} = \begin{pmatrix}4\\t+4\\t+3\\-3\end{pmatrix}.$$

Calculate the standard matrix of T. For what values of t are the columns of the standard matrix linearly dependent? For what values of t is the span of the columns $\mathbb{R}^4$? Justify your answers.

Solution:

$$T\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = T\begin{pmatrix}1\\-1\\0\\0\end{pmatrix} - T\begin{pmatrix}0\\-1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\-t\\0\\t\end{pmatrix} - \begin{pmatrix}-1\\-t\\-1\\t\end{pmatrix} = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = -T\begin{pmatrix}0\\-1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\t\\1\\-t\end{pmatrix}$$

$$T\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = -T\begin{pmatrix}0\\0\\-1\\0\end{pmatrix} = \begin{pmatrix}2\\3\\t+1\\-3\end{pmatrix}$$

$$T\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = T\begin{pmatrix}0\\1\\1\\1\end{pmatrix} + T\begin{pmatrix}0\\-1\\0\\0\end{pmatrix} + T\begin{pmatrix}0\\0\\-1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\\t\end{pmatrix}$$

$$\Rightarrow \text{Standard Matrix} = \begin{pmatrix}1&1&2&1\\0&t&3&1\\1&1&t+1&1\\0&-t&-3&t\end{pmatrix}$$

$$\begin{pmatrix}1&1&2&1\\0&t&3&1\\1&1&t+1&1\\0&-t&-3&t\end{pmatrix} \longrightarrow \begin{pmatrix}1&1&2&1\\0&t&3&1\\0&0&t-1&0\\0&-t&-3&t\end{pmatrix} \longrightarrow \begin{pmatrix}\boxed{1}&1&2&1\\0&\boxed{t}&3&1\\0&0&\boxed{t-1}&0\\0&0&0&\boxed{t+1}\end{pmatrix}$$

$$\Rightarrow \text{Columns L.D.} \iff t = 0, 1, -1$$

$$\text{Span columns} = \mathbb{R}^4 \iff t \neq 0, 1, -1$$

END OF EXAM