MATH 54 MIDTERM 1 (001)

PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.
CALCULATORS ARE NOT PERMITTED
YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK
SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE
REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT
THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name and section: ____

GSI's name: ____

This exam consists of 5 questions. Answer the questions in the spaces provided.

1. (25 points) (a) Calculate the general solution to the linear system with the following augmented matrix:

$$\left(\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Solution:

$$\left(\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 2 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & 0 & -6 \\ 0 & 0 & 1 & 2 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{array}{l} x_1 + 2x_4 = -6 \\ x_3 + 2x_4 = -4 \\ x_5 = 3 \end{array} \Rightarrow \text{General solution is } \begin{pmatrix} -6 - 2x_4 \\ x_2 \\ -4 - 2x_4 \\ x_4 \\ 3 \end{pmatrix}$$

where x_2, x_4 are free

(b) Will the above coefficient matrix always give a consistent linear system? Justify your answer.

Solution:

No, if $\underline{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ then the final column is a pivot, hence the system is inconsistent

2. (25 points) Calculate the determinant and inverse matrix of $\begin{pmatrix} 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$.

Solution:

Swith 1st and 4th

$$\left(\begin{array}{cccc|cccc} 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

$$\text{determinant} = (-1) \cdot 1 \cdot 1 \cdot (-1) \cdot (-1) = -1$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -2 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -2 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 2 & 0 \end{array}\right)$$

Inverse

3. (25 points) Find all possible value of a, b, c such that $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is a solution to homogeneous linear system

$$\left(\begin{array}{ccc|c} a & b & c-1 & 0 \\ b & c & c+2 & 0 \\ 2c & -b & a & 0 \end{array}\right)$$

Solution:

$$\begin{aligned} a + b - (c-1) &= 0 \\ b + c - (c+2) &= 0 \\ 2c - b - a &= 0 \end{aligned} \Rightarrow \begin{aligned} a + b - c &= -1 \\ b &= 2 \\ -a - b + 2c &= 0 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 2 \\ -1 & -1 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \longleftarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$\Rightarrow$ $a = -4$, $b = 2$, $c = -1$

4. (25 points) (a) Let A be a 4×5 matrix with the following properties:

The second column is non-zero and is a scalar multiple of the first. The third column is not a scalar multiple of the first.

Write the echelon form matrices which are potentially row equivalent to A.

Solution:

$$\begin{pmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & \blacksquare \end{pmatrix}$$

(b) Let two matrices A and B satisfy the above conditions. If T_A and T_B are both onto must A and B be row equivalent? Justify your answer.

Solution:

No : $A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Different Reduced echelon matrices

A and B row equivalent $\Longleftrightarrow$ exactly same reduced echelon matrix

5. (25 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation such that

$$T\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\t+1\\t+2\end{pmatrix},\ T\begin{pmatrix}0\\2\\2\end{pmatrix} = \begin{pmatrix}0\\2t+2\\2t+2\\4t+4\end{pmatrix},\ T\begin{pmatrix}0\\-1\\0\end{pmatrix} = \begin{pmatrix}-1\\-t\\-1\\-t\end{pmatrix}.$$

Calculate the standard matrix of T. For what values of t is T one-to-one? For what value of t is T onto? Justify your answer.

Solution:

$$T\begin{pmatrix}0\\1\\1\end{pmatrix} = \frac{1}{2}T\begin{pmatrix}0\\2\\2\end{pmatrix} = \begin{pmatrix}0\\t+1\\t+1\\2t+2\end{pmatrix}$$

$$T\begin{pmatrix}0\\-1\\0\end{pmatrix} = \begin{pmatrix}-1\\-t\\-1\\-t\end{pmatrix} \Rightarrow T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\t\\1\\t\end{pmatrix}$$

$$\Rightarrow T\begin{pmatrix}0\\0\\1\end{pmatrix} = T\begin{pmatrix}0\\1\\1\end{pmatrix} - T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\\t\\t+2\end{pmatrix}$$

$$\Rightarrow T\begin{pmatrix}1\\0\\0\end{pmatrix} = T\begin{pmatrix}1\\0\\1\end{pmatrix} - T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$\Rightarrow$ Standard Matrix is $\begin{pmatrix}1 & 1 & -1\\0 & t & 1\\1 & 1 & t\\0 & t & t+2\end{pmatrix}$

$$\begin{pmatrix}1 & 1 & -1\\0 & t & 1\\1 & 1 & t\\0 & t & t+2\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 1 & -1\\0 & t & 1\\0 & 0 & t+1\\0 & t & t+2\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 1 & -1\\0 & t & 1\\0 & 0 & t+1\\0 & 0 & t+1\end{pmatrix}$$

$$\downarrow$$

T one-to-one if and only if $\Leftarrow \begin{pmatrix}\boxed{1} & 1 & -1\\0 & \boxed{t} & 1\\0 & 0 & \boxed{t+1}\\0 & 0 & 0\end{pmatrix}$

$t \neq 0$ <u>and</u> $t \neq -1$

T <u>never</u> onto. Never Pivot in last row.

END OF EXAM