MATH 54 FINAL EXAM (PRACTICE 3)

PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name and section: ______________________________

GSI's name: ______________________________

This exam consists of 10 questions. Answer the questions in the spaces provided.

1. (25 points) Find all possible values of a, b, c such that $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ is a solution to linear system

$$\left(\begin{array}{ccc|c} a+1 & b & 0 & c \\ 0 & c & a & 2 \\ a+b & -1 & -c & 0 \end{array}\right)$$

Solution:

$$\begin{aligned} 1\cdot(a+1) - b + 2\cdot 0 &= c \\ 1\cdot 0 - c + 2a &= 2 \\ 1\cdot(a+b) + 1 - 2c &= 0 \end{aligned} \implies \begin{aligned} a - b - c &= -1 \\ 2a - c &= 2 \\ a + b - 2c &= -1 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 2 & 0 & -1 & 2 \\ 1 & 1 & -2 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 0 & 2 & 1 & 4 \\ 0 & 2 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & -2 & -4 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right) \leftarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\implies a = 2,\ b = 1,\ c = 2$$

2. (25 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a one-to-one linear transformation such that

$$T(\underline{\mathbf{x}}) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \end{pmatrix}, T(\underline{\mathbf{y}}) = \begin{pmatrix} 0 \\ -1 \\ 4 \\ -1 \end{pmatrix}, T(\underline{\mathbf{z}}) = \begin{pmatrix} 2 \\ 2 \\ 14 \\ -4 \end{pmatrix}.$$

Is it possible for the vectors $\{\underline{\mathbf{x}}, \underline{\mathbf{y}}, \underline{\mathbf{z}}\}$ to be linearly independent? Is it possible for T to be onto? Justify your answers.

Solution:

Observe that $2\begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 0 \\ -1 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 14 \\ -4 \end{pmatrix}$

$\Rightarrow \quad 2\begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 0 \\ -1 \\ 4 \\ -1 \end{pmatrix} + (-1)\begin{pmatrix} 2 \\ 2 \\ 14 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \quad T(2\underline{x} + 2\underline{y} + (-1)\underline{z}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \quad 2\underline{x} + 2\underline{y} + (-1)\underline{z} = \underline{0}$ $\quad$ (T one-to-one and $T(\underline{0}) = \underline{0}$)

$\Rightarrow \quad \{\underline{x}, \underline{y}, \underline{z}\}$ L.D.

$T = T_A$ for some A a 4×3 matrix.

$\Rightarrow$ Number of pivot positions ≤ 3 $\Rightarrow$ Number of pivot positions $< 4 =$ number of rows $\Rightarrow$ T_A not onto.

3. (25 points) (a) Let V be a vector space. Carefully define what it means for a subset $U \subset V$ to be a subspace.

Solution:

$U \subset V$ is a subspace if

1/ $\underline{0}_V$ is contained in U

2/ If $\underline{u}, \underline{v}$ are in U, then $\underline{u} + \underline{v}$ is in U

3/ If $\underline{u}$ is in U and λ is in $\mathbb{R}$, then $\lambda \underline{u}$ is in U

(b) Let V be the vector space of continuous real-valued functions on the closed interval $[0, 1]$. Let U be the subset of V consisting of those functions f such that $f(0) \leq f(1)$. Is U a subspace? Carefully justify your answer.

Solution:

It is not a subspace.

Conditions 1/ and 2/ are satisfied. However 3/ is not.

E.g. $f(x) = x$ is in U because $f(0) = 0 \leq 1 = f(1)$

However $(-1)f(x) = -x$ and $(-1)f(0) = 0 \not\leq -1 = (-1)f(1)$

4. (25 points) Let M_2 be the vector space of 2×2 matrices with real entries. Let T be the following linear transformation:

$$\begin{aligned} T : M_2 &\to M_2 \\ A &\mapsto A - A^T \end{aligned}$$

Find a basis for $Ker(T)$. What is $Rank(T)$?

Solution:

$$\begin{aligned} Ker(T) &= \{ A \text{ in } M_2 \text{ such that } T(A) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \} \\ &= \{ A \text{ in } M_2 \text{ such that } A - A^T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \} \\ &= \{ A \text{ in } M_2 \text{ such that } A = A^T \} \leftarrow \text{symmetric } 2\times 2 \text{ matrices.} \end{aligned}$$

$$\begin{aligned} \Rightarrow Ker(T) &= \{ \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \text{ where } a, b, c \text{ real} \} \\ &= \{ a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \text{ where } a, b, c \text{ real} \} \\ &= \text{Span}\left(\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{Basis for } Ker(T)} \} \right) \end{aligned}$$

$$Dim(M_2) = 4$$

$$\text{Rank-Nullity} \Rightarrow \overbrace{\dim(Range(T))}^{\text{Rank}} + \overbrace{\dim(Ker(T))}^{\text{Nullity}} = \dim(M_2)$$

$$\Rightarrow Rank(T) = 1$$

5. (25 points) Let T be the following linear transformation:

$$\begin{aligned} T : \mathbb{P}_2(\mathbb{R}) &\rightarrow \mathbb{P}_3(\mathbb{R}) \\ p(x) &\mapsto p'(x) + p(x) \end{aligned}$$

Find bases B and C, for $\mathbb{P}_2(\mathbb{R})$ and $\mathbb{P}_3(\mathbb{R})$ respectively, such that

$$A_{B,C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solution:

Let $\mathcal{B} = \{1, x, x^2\}$, the standard basis for $\mathbb{P}_2(\mathbb{R})$

$T(1) = 1$

$T(x) = 1 + x$

$T(x^2) = 2x + x^2$

Note that $c_1 1 + c_2(1+x) + c_3(2x + x^2) = 0 \Rightarrow c_1 = 0, c_2 = 0, c_3 = 0$

$\Rightarrow \{1, 1+x, 2x+x^2\}$ L.I.

Extend to a basis by including x^3

$\mathcal{C} = \{1, 1+x, 2x+x^2, x^3\}$

$(T(1))_{\mathcal{C}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $(T(x))_{\mathcal{C}} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $(T(x^2))_{\mathcal{C}} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow A_{\mathcal{B},\mathcal{C}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

6. (25 points) Let W be the span of the vectors $\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\-2\\0\\1\end{pmatrix}$ in $\mathbb{R}^4$. Find two orthogonal vectors, $\underline{\mathbf{u}}, \underline{\mathbf{v}}$, such $\underline{\mathbf{u}} + \underline{\mathbf{v}} = \begin{pmatrix}1\\0\\1\\-1\end{pmatrix}$ and $\underline{\mathbf{u}}$ is in W?

Solution:

$$\underline{v}_1 = \begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$\underline{v}_2 = \begin{pmatrix}0\\-2\\0\\1\end{pmatrix} - \frac{\begin{pmatrix}0\\-2\\0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\0\\1\end{pmatrix}}{\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\0\\1\end{pmatrix}} \begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \begin{pmatrix}-1/2\\-2\\0\\1/2\end{pmatrix}$$

Orthogonal basis

$$\Rightarrow \quad W = \operatorname{Span}\left\{ \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\-4\\0\\1\end{pmatrix} \right\}$$

$$\Rightarrow \quad \operatorname{Proj}_W \begin{pmatrix}1\\0\\1\\-1\end{pmatrix} = \frac{\begin{pmatrix}1\\0\\1\\-1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\0\\1\end{pmatrix}}{\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\0\\1\end{pmatrix}} \begin{pmatrix}1\\0\\0\\1\end{pmatrix} + \frac{\begin{pmatrix}1\\0\\1\\-1\end{pmatrix} \cdot \begin{pmatrix}-1\\-4\\0\\1\end{pmatrix}}{\begin{pmatrix}-1\\-4\\0\\1\end{pmatrix} \cdot \begin{pmatrix}-1\\-4\\0\\1\end{pmatrix}} \begin{pmatrix}-1\\-4\\0\\1\end{pmatrix}$$

$$= \frac{-2}{18} \begin{pmatrix}-1\\-4\\0\\1\end{pmatrix} = \begin{pmatrix}1/9\\4/9\\0\\-1/9\end{pmatrix} = \underline{u}$$

$$\underline{v} = \begin{pmatrix}1\\0\\1\\-1\end{pmatrix} - \begin{pmatrix}1/9\\4/9\\0\\-1/9\end{pmatrix} = \begin{pmatrix}8/9\\-4/9\\1\\-8/9\end{pmatrix}$$

7. (25 points) Give a singular-value decomposition of the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Solution:

$$A^T A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$\Rightarrow$ eigenvalues of $A^T A$ are $4, 1, 0$

$\Rightarrow$ Singular-values are $2, 1, 0$

Set $\underline{v}_1 = \underline{e}_3$, $\underline{v}_2 = \underline{e}_1$, $\underline{v}_3 = \underline{e}_2$

$$\underline{u}_1 = \frac{1}{2} A \underline{v}_1 = \frac{1}{2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \underline{e}_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\underline{u}_2 = \frac{1}{1} A \underline{v}_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \underline{e}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Solution (continued) :

8. (25 points) Find a solution to the following initial value problem

$$y'' + 2y' + 2y = e^t\cos(t), \quad y(0) = 0, y'(0) = 1$$

Solution:

$$r^2 + 2r + 2 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$\Rightarrow$ General Solution to $y'' + 2y' + 2y$ $= c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)$

$$y_p(t) = A_0 e^t \cos(t) + B_0 e^t \sin(t)$$

$$\Rightarrow y_p'(t) = A_0 e^t \cos(t) - A_0 e^t \sin(t) + B_0 e^t \sin(t) + B_0 e^t \cos(t)$$

$$\Rightarrow y_p''(t) = A_0 e^t \cos(t) - A_0 e^t \sin(t) - A_0 e^t \sin(t) - A_0 e^t \cos(t)$$
$$+ B_0 e^t \sin(t) + B_0 e^t \cos(t) + B_0 e^t \cos(t) - B_0 e^t \sin(t)$$
$$= 2B_0 e^t \cos(t) - 2A_0 e^t \sin(t)$$

$$\Rightarrow y_p''(t) + 2y_p'(t) + 2y_p(t)$$
$$= 2B_0 e^t \cos(t) - 2A_0 e^t \sin(t)$$
$$+ 2\left(A_0 e^t \cos(t) - A_0 e^t \sin(t) + B_0 e^t \sin(t) + B_0 e^t \cos(t)\right)$$
$$+ 2\left(A_0 e^t \cos(t) + B_0 e^t \sin(t)\right)$$

$$= (4A_0 + 4B_0) e^t \cos(t) + (-4A_0 + 4B_0) e^t \sin(t)$$

Solution (continued) :

$$\Rightarrow \quad \begin{aligned} 4A_0 + 4B_0 &= 1 \\ -4A_0 + 4B_0 &= 0 \end{aligned} \quad \Rightarrow \quad A_0 = B_0 = \frac{1}{8}$$

$\Rightarrow$ General Solution to $y'' + 2y' + 2y = e^t \cos(t)$

$$= \frac{1}{8} e^t \cos(t) + \frac{1}{8} e^t \sin(t) + c_1 e^{-t} \cos(t) + c_2 e^{-t} \sin(t)$$

$$y(0) = \frac{1}{8} + c_1 = 0$$

$$y'(t) = \frac{1}{8} e^t \cos(t) - \frac{1}{8} e^t \sin(t) + \frac{1}{8} e^t \sin(t) + \frac{1}{8} e^t \cos(t)$$
$$- c_1 e^{-t} \cos(t) - c_1 e^{-t} \sin(t) - c_2 e^{-t} \sin(t) + c_2 e^{-t} \cos(t)$$

$$\Rightarrow y(0) = \frac{1}{4} - c_1 + c_2 = 1$$

$$\Rightarrow \quad c_1 = \frac{-1}{8}, \quad c_2 = \frac{5}{8}$$

$$\Rightarrow \quad y(t) = \frac{1}{8} e^t \cos(t) + \frac{1}{8} e^t \sin(t) + \left(\frac{-1}{8}\right) e^{-t} \cos(t) + \left(\frac{5}{8}\right) e^{-t} \sin(t)$$

9. (25 points) Find a solution to the initial value problem

$$\underline{x}'(t) = \overbrace{\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}}^{=A} \underline{x}(t), \qquad \underline{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solution:

$$\det(A - x I_2) = (1-x)^2 + 4 = 0 \Rightarrow 1 - x = \pm 2i \Rightarrow x = 1 \pm 2i$$

$$\mathrm{Nul}(A - (1+2i) I_2) = \mathrm{Nul}\begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix}$$

$$\frac{1}{-2i} = \frac{1}{2} i$$

$$\begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix} \Rightarrow \mathrm{Nul}\begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} = \left\{ \begin{pmatrix} i x_2 \\ x_2 \end{pmatrix} \right\}$$

$$= \mathrm{Span}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \text{General Solution} = c_1\left(e^t \cos(2t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} - e^t \sin(2t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + c_2\left(e^t \sin(2t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^t \cos(2t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$$\underline{x}(0) = c_1\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + c_2\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} c_1 = 0 \\ c_2 = 1 \end{matrix}$$

$$\Rightarrow \underline{x}(t) = e^t \sin(2t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^t \cos(2t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

10. (25 points) Calculate the Fourier series of the function $f(x) = \begin{cases} 1 & \pi/2 \le x \le \pi \\ 0 & -\pi \le x < \pi/2 \end{cases}$, on the interval $[\pi, \pi]$. What doe the Fourier series converge to at $x = 7\pi/2$?

Solution:

$n \geq 1$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{\pi/2}^{\pi} \cos(nx)\,dx = \frac{1}{n\pi}\sin(nx)\Big|_{\pi/2}^{\pi}$$

$$= \frac{-1}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$n = 0$

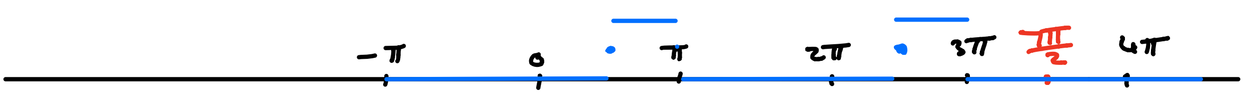
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{\pi/2}^{\pi} 1\,dx = \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{\pi/2}^{\pi}\sin(nx)\,dx = \frac{-1}{n\pi}\cos(nx)\Big|_{\pi/2}^{\pi}$$

$$= \frac{-1}{n\pi}\left((-1)^n - \cos\left(\frac{n\pi}{2}\right)\right)$$

$$\Rightarrow F.S. = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)$$

$$= \frac{1}{4} + \sum_{n=1}^{\infty}\left(\frac{-1}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos(nx) + \frac{-1}{n\pi}\left((-1)^n - \cos\left(\frac{n\pi}{2}\right)\right)\sin(nx)\right)$$



$\Rightarrow$ F.S. converges to 0 at $x = \frac{7\pi}{2}$