MATH 54 FINAL EXAM (PRACTICE 3)

PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name and section: ____

GSI's name: ____

This exam consists of 10 questions. Answer the questions in the spaces provided.

1. (25 points) Find all possible values of a, b, c such that $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ is a solution to linear system

$$\left(\begin{array}{ccc|c} a+1 & b & 0 & c \\ 0 & c & a & 2 \\ a+b & -1 & -c & 0 \end{array}\right)$$

Solution:

2. (25 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a one-to-one linear transformation such that

$$T(\underline{\mathbf{x}}) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \end{pmatrix}, T(\underline{\mathbf{y}}) = \begin{pmatrix} 0 \\ -1 \\ 4 \\ -1 \end{pmatrix}, T(\underline{\mathbf{z}}) = \begin{pmatrix} 2 \\ 2 \\ 14 \\ -4 \end{pmatrix}.$$

Is it possible for the vectors $\{\underline{\mathbf{x}}, \underline{\mathbf{y}}, \underline{\mathbf{z}}\}$ to be linearly independent? Is it possible for T to be onto? Justify your answers.

Solution:

3. (25 points) (a) Let V be a vector space. Carefully define what it means for a subset $U \subset V$ to be a subspace.

   Solution:

   (b) Let V be the vector space of continuous real-valued functions on the closed interval $[0,1]$. Let U be the subset of V consisting of those functions f such that $f(0) \leq f(1)$. Is U a subspace? Carefully justify your answer.

   Solution:

4. (25 points) Let M_2 be the vector space of 2×2 matrices with real entries. Let T be the following linear transformation:

$$\begin{aligned} T : M_2 &\rightarrow M_2 \\ A &\mapsto A - A^T \end{aligned}$$

Find a basis for $Ker(T)$. What is $Rank(T)$?

Solution:

5. (25 points) Let T be the following linear transformation:

$$\begin{aligned} T : \mathbb{P}_2(\mathbb{R}) &\to \mathbb{P}_3(\mathbb{R}) \\ p(x) &\mapsto p'(x) + p(x) \end{aligned}$$

Find bases B and C, for $\mathbb{P}_2(\mathbb{R})$ and $\mathbb{P}_3(\mathbb{R})$ respectively, such that

$$A_{B,C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solution:

6. (25 points) Let W be the span of the vectors $\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\-2\\0\\1\end{pmatrix}$ in $\mathbb{R}^4$. Find two orthogonal vectors, $\underline{\mathbf{u}}, \underline{\mathbf{v}}$, such $\underline{\mathbf{u}} + \underline{\mathbf{v}} = \begin{pmatrix}1\\0\\1\\-1\end{pmatrix}$ and $\underline{\mathbf{u}}$ is in W?

Solution:

7. (25 points) Give a singular-value decomposition of the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Solution:

Solution (continued) :

8. (25 points) Find a solution to the following initial value problem

$$y'' + 2y' + 2y = e^t \cos(t), \quad y(0) = 0, y'(0) = 1$$

Solution:

Solution (continued) :

9. (25 points) Find a solution to the initial value problem

$$\underline{x}'(t) = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \underline{x}(t), \qquad \underline{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solution:

10. (25 points) Calculate the Fourier series of the function $f(x) = \begin{cases} 1 & \pi/2 \leq x \leq \pi \\ 0 & -\pi \leq x < \pi/2 \end{cases}$, on the interval $[\pi, \pi]$. What doe the Fourier series converge to at $x = 7\pi/2$?

Solution:

END OF EXAM