MATH 54 FINAL EXAM (PRACTICE 2)

PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name and section: ______________________________

GSI's name: ______________________________

This exam consists of 10 questions. Answer the questions in the spaces provided.

1. (25 points) (a) Are the following matrices row equivalent?

$$\begin{pmatrix} 2 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & -2 & 2 \\ 2 & 0 & 4 & 3 \end{pmatrix}$$

Solution:

Reduced Echelon Form

$$\begin{pmatrix} 2 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -1 & -3 & 1 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & -2 & 2 \\ 2 & 0 & 4 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

$\Rightarrow \begin{pmatrix} 2 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & -2 & 2 \\ 2 & 0 & 4 & 3 \end{pmatrix}$ are not row equivalent

(b) What are the dimensions of the null and column spaces of the above matrices?

Solution:

dimension of column space = number of pivot columns = 3

dimension of null space = number of free columns = 1

2. (25 points) Do the following vectors span $\mathbb{R}^3$?

$$\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}2\\2\\-1\end{pmatrix}, \begin{pmatrix}3\\2\\-2\end{pmatrix}$$

If a matrix has three of these vectors as columns, can it be invertible? Carefully justify your answers.

Solution:

$$\begin{pmatrix}1 & 1 & 2 & 3\\2 & 0 & 2 & 2\\0 & -1 & -1 & -2\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 1 & 2 & 3\\0 & -2 & -2 & -4\\0 & -1 & -1 & -2\end{pmatrix} \longrightarrow \begin{pmatrix}\boxed{1} & 1 & 2 & 3\\0 & \boxed{1} & 1 & 2\\0 & 0 & 0 & 0\end{pmatrix}$$

$$\Rightarrow \quad \text{Span}\left(\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}2\\2\\-1\end{pmatrix}, \begin{pmatrix}3\\2\\-2\end{pmatrix}\right) = \text{Span}\left(\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}\right)$$

and $\quad \dim\left(\text{Span}\left(\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}\right)\right) = 2$

$\dim(\mathbb{R}^3) = 3$

$$\Rightarrow \quad \text{Span}\left(\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}\right) \neq \mathbb{R}^3$$

If A is a 3×3 matrix with columns given by 3 of these vectors $\text{Rank}(A) < 3$. Hence A <u>not</u> invertible.

3. (25 points) (a) Let $T : V \to W$ be a linear transformation between two vector spaces. Define the kernel of T, $Ker(T)$. Show that $Ker(T)$ is a subspace of V. You may assume that $T(\underline{0}_V) = \underline{0}_W$

Solution:

$Ker(T) = \{ \underline{v}$ in V such that $T(\underline{v}) = \underline{0}_W \}$

Claim : $Ker(T) \subset V$ is a subspace

Proof

1/ $T(\underline{0}_V) = \underline{0}_W \Rightarrow \underline{0}_V$ in $Ker(T)$

2/ $\underline{u}, \underline{v}$ in $Ker(T) \Rightarrow T(\underline{u}) = \underline{0}_W, T(\underline{v}) = \underline{0}_W$

$\Rightarrow T(\underline{u}) + T(\underline{v}) = \underline{0}_W + \underline{0}_W = \underline{0}_W$

$\Rightarrow T(\underline{u}+\underline{v}) = \underline{0}_W \Rightarrow \underline{u}+\underline{v}$ in $Ker(T)$

3/ $\underline{u}$ in $Ker(T)$, λ in $\mathbb{R} \Rightarrow T(\underline{u}) = \underline{0}_W$

$\Rightarrow \lambda T(\underline{u}) = \lambda \underline{0}_W = \underline{0}_W$

$\Rightarrow T(\lambda \underline{u}) = \underline{0}_W \Rightarrow \lambda \underline{u}$ in $Ker(T)$

$\square$

(b) Does there exist a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, such that $Ker(T) = \{\begin{pmatrix} x \\ x+1 \end{pmatrix}\}$ where x is any real number?

Solution:

No! $\begin{pmatrix} x \\ x+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = 0$ and $x = -1$.

$\Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not in the set $\{ \begin{pmatrix} x \\ x+1 \end{pmatrix}, x \text{ real} \}$. Hence it is <u>not</u> a subspace.

4. (25 points) Let T be the following linear transformation:

$$\begin{aligned} T : \mathbb{P}_2(\mathbb{R}) &\to \mathbb{P}_2(\mathbb{R}) \\ p(x) &\mapsto p'(x) + p(x) \end{aligned}$$

Does there exist a basis $B \subset \mathbb{P}_2(\mathbb{R})$ such that $A_{B,B}$ is diagonal? Justify your answer. Hint: Think about the possible degrees of the polynomials in B.

Solution:

$$\beta = \{p_1(x), p_2(x), p_3(x)\}$$

$$A_{\beta,\beta} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \iff \begin{aligned} T(p_1(x)) &= \lambda_1 p_1(x) \\ T(p_2(x)) &= \lambda_2 p_2(x) \\ T(p_3(x)) &= \lambda_3 p_3(x) \end{aligned}$$

$p(x) \neq 0$

$$T(p(x)) = \lambda p(x) \iff p'(x) + p(x) = \lambda p(x)$$

$$\iff (\lambda - 1) p(x) = p'(x) \qquad \text{degree}(p'(x)) < \text{degree}(p(x))$$

so the only way this can happen is iff $\lambda - 1 = 0$

and $p(x)$ constant.

But $\{p_1(x), p_2(x), p_3(x)\}$ are a basis for $\mathbb{P}_2(\mathbb{R})$

$\Rightarrow$ No such basis exists.

5. (25 points) Let $C[-1,1]$ be the inner product space of real-valued functions on the closed interval $[-1,1]$, such that

$$< f, g >= \int_{-1}^{1} f(x)g(x)dx.$$

Find an orthogonal basis for $W = Span(1, x^2, x^4)$.

Solution:

Must apply Gram-Schmidt :

$$\underline{v_1} = 1$$

$$\underline{v_2} = x^2 - \frac{<x^2, 1>}{<1, 1>} 1 = x^2 - \frac{\int_{-1}^{1} x^2 dx}{\int_{-1}^{1} 1\, dx} 1 = x^2 - \frac{\frac{2}{3}}{2} \cdot 1$$

$$= x^2 - \frac{1}{3}$$

$$\underline{v_3} = x^4 - \frac{<x^4, 1>}{<1, 1>} 1 - \frac{<x^4, x^2 - \frac{1}{3}>}{<x^2 - \frac{1}{3}, x^2 - \frac{1}{3}>} (x^2 - \tfrac{1}{3})$$

$$\int_{-1}^{1} x^4 dx = 2\int_{0}^{1} x^4 dx = \frac{2}{5} \quad , \quad \int_{-1}^{1} 1\, dx = 2$$

$$\int_{-1}^{1} x^4 (x^2 - \tfrac{1}{3}) dx = 2\int_{0}^{1} x^6 - \tfrac{1}{3} x^4 dx = 2\left(\tfrac{1}{7} - \tfrac{1}{15}\right)$$

$$\int_{-1}^{1} (x^2 - \tfrac{1}{3})^2 dx = \int_{-1}^{1} x^4 - \tfrac{2}{3} x^2 + \tfrac{1}{9}\, dx = 2\int_{0}^{1} x^4 - \tfrac{2}{3} x^2 + \tfrac{1}{9}\, dx$$

$$= 2\left(\tfrac{1}{5} - \tfrac{2}{9} + \tfrac{1}{9}\right)$$

$$\Rightarrow \quad \underline{v_3} = x^4 - \frac{\frac{2}{5}}{2} 1 - \frac{2\left(\frac{1}{7} - \frac{1}{15}\right)}{2\left(\frac{1}{5} - \frac{1}{9}\right)} (x^2 - \tfrac{1}{3}) = x^4 - \tfrac{90}{105} x^2 + \tfrac{9}{105}$$

$\Rightarrow \quad \left\{ 1, \; x^2 - \frac{1}{3}, \; x^4 - \frac{90}{105} x^2 + \frac{9}{105} \right\}$ is an orthogonal basis

6. (25 points) Determine all least-squares solutions to the following linear system:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \underline{x} = \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 14 \\ 4 \\ 10 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 4 & 2 & 2 & 14 \\ 2 & 2 & 0 & 4 \\ 2 & 0 & 2 & 10 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 2 & 0 & 4 \\ 4 & 2 & 2 & 14 \\ 2 & 0 & 2 & 10 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 2 & 0 & 4 \\ 0 & -2 & 2 & 6 \\ 0 & -2 & 2 & 6 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 2 & 0 & 4 \\ 0 & -2 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 2 & 2 & 0 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 0 & 2 & 10 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ General Least-Squares solution $= \left\{ \begin{pmatrix} 5 - x_3 \\ -3 + x_3 \\ x_3 \end{pmatrix}, \ x_3 \text{ real} \right\}$

7. (25 points) Find orthonormal bases of $\mathbb{R}^2$, B and C, such that if

$$A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$$

then $A_{B,C}$ is diagonal with non-negative entries.

Solution:

$$A^TA = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} 20 & -10 \\ -10 & 5 \end{pmatrix}$$

$$\det(A^TA - x I_2) = (20-x)(5-x) - 100 = x^2 - 25x$$

$\Rightarrow$ eigenvalues of A^TA are 25 and 0

$\Rightarrow$ Singular-values of A are 5 and 0

$$\mathrm{Nul}(A^TA - 25 I_2) = \mathrm{Nul}\begin{pmatrix} -5 & -10 \\ -10 & -20 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \mathrm{Span}\left(\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right)$$

$$\mathrm{Nul}(A^TA - 0 I_2) = \mathrm{Nul}\begin{pmatrix} 20 & -10 \\ -10 & 5 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$$

$$\left\|\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\| = \sqrt{5}, \quad \left\|\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\| = \sqrt{5}$$

Let $\underline{v}_1 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$

$$\underline{u}_1 = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -5/\sqrt{5} \\ 10/\sqrt{5} \end{pmatrix} = \begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

Solution (continued) :

$$Nul(A^T) = Nul\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} = Nul\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} = Span\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$$

$$\left\|\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\| = \sqrt{5}$$

Let $\underline{u}_2 = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$

Orthonormal bases

Let $\mathcal{B} = \left\{\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}\right\}$, $\mathcal{C} = \left\{\begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}, \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}\right\}$

$$\Rightarrow \quad A_{\mathcal{B},\mathcal{C}} = \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}$$

8. (25 points) Find a general solution to the following differential equation

$$y'' - y = t\cos(t) + \sin(t)$$

Solution:

$r^2 - 1 = 0 \Rightarrow r = \pm 1 \Rightarrow$ General solution to $y'' - y = 0$ is $c_1 e^t + c_2 e^{-t}$

$y_p(t) = (A_0 + A_1 t)\cos(t) + (B_0 + B_1 t)\sin(t)$

$\Rightarrow y_p'(t) = A_1 \cos(t) - (A_0 + A_1 t)\sin(t) + B_1 \sin(t) + (B_0 + B_1 t)\cos(t)$

$= (A_1 + B_0 + B_1 t)\cos(t) + (B_1 - A_0 - A_1 t)\sin(t)$

$\Rightarrow y_p''(t) = B_1 \cos(t) - (A_1 + B_0 + B_1 t)\sin(t)$

$- A_1 \sin(t) + (B_1 - A_0 - A_1 t)\cos(t)$

$= (2B_1 - A_0 - A_1 t)\cos(t) + (-2A_1 - B_0 - B_1 t)\sin(t)$

$\Rightarrow y_p''(t) - y_p(t) = (2B_1 - A_0 - A_1 t)\cos(t) + (-2A_1 - B_0 - B_1 t)\sin(t)$

$- (A_0 + A_1 t)\cos(t) - (B_0 + B_1 t)\sin(t)$

$= (2B_1 - 2A_0 - 2A_1 t)\cos(t) + (-2A_1 - 2B_0 - 2B_1 t)\sin(t)$

$= t\cos(t) + \sin(t)$

$\Rightarrow$ $2B_1 - 2A_0 = 0$, $-2A_1 - 2B_0 = 1$

$-2A_1 = 1$, $-2B_1 = 0$

Solution (continued) :

$\Rightarrow \quad B_1 = 0 \quad \Rightarrow \quad A_0 = 0$

$A_1 = \frac{-1}{2} \quad \Rightarrow \quad B_0 = 0$

$\Rightarrow$ General Solution to

$$y'' - y = t\cos(t) + \sin(t) \quad = \quad \frac{-1}{2} t\cos(t) + c_1 e^{t} + c_2 e^{-t}$$

9. (25 points) Find a general solution to

$$\underline{x}'(t) = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}}_{= A} \underline{x}(t).$$

Solution:

$$\det(A - xI_3) = \det\begin{pmatrix} 1-x & 0 & 0 \\ 0 & 1-x & 1 \\ 0 & -1 & 1-x \end{pmatrix} = (1-x)\left((1-x)^2+1\right) = 0$$

$$\Rightarrow \quad x = 1 \text{ or } 1-x = \pm i \Rightarrow x = 1 \text{ or } x = 1 \pm i$$

$$\mathrm{Nul}(A - 1\cdot I_3) = \mathrm{Nul}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\mathrm{Nul}(A - (1+i)I_3) = \mathrm{Nul}\begin{pmatrix} -i & 0 & 0 \\ 0 & -i & 1 \\ 0 & -1 & -i \end{pmatrix}$$

$$\left(\frac{1}{-i} = i\right)$$

$$\begin{pmatrix} -i & 0 & 0 \\ 0 & -i & 1 \\ 0 & -1 & -i \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & -1 & -i \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \mathrm{Nul}(A - (1+i)I_3) = \left\{\begin{pmatrix} 0 \\ -ix_3 \\ x_3 \end{pmatrix}\right\} = \mathrm{Span}\left(\begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}\right)$$

$$\begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad e^t\cos(t)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - e^t\sin(t)\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad e^t\sin(t)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + e^t\cos(t)\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

are L.I. solutions

Solution (continued) :

$$\Rightarrow \text{General Solution} = c_1 e^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$+ c_2 \left(e^{t} \cos(t) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - e^{t} \sin(t) \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right)$$

$$+ c_3 \left(e^{t} \sin(t) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + e^{t} \cos(t) \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right)$$

10. (25 points) Calculate the cosine Fourier series of the function $f(x) = e^x$, on the interval $[0, \pi]$.

Solution:

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^x \cos(nx)\, dx$$

$$\int e^x \cos(nx)\,dx = e^x \cdot \frac{1}{n}\sin(nx) - \int e^x \frac{1}{n}\sin(nx)\,dx$$

$$\int e^x \sin(nx)\,dx = e^x\left(\frac{-1}{n}\right)\cos(nx) + \int e^x \frac{1}{n}\cos(nx)\,dx$$

$$\Rightarrow \int e^x \cos(nx)\,dx = e^x \cdot \frac{1}{n}\sin(nx) + e^x\left(\frac{1}{n^2}\right)\cos(nx) - \frac{1}{n^2}\int e^x \cos(nx)\,dx$$

$$\Rightarrow \int e^x \cos(nx)\,dx = \frac{1}{1+\frac{1}{n^2}} e^x\left(\frac{1}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right)$$

$$\Rightarrow \int_0^{\pi} e^x \cos(nx)\,dx = \frac{1}{1+\frac{1}{n^2}} e^x\left(\frac{1}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right)\Bigg|_0^{\pi}$$

$$= \left(\frac{1}{1+\frac{1}{n^2}}\right)e^{\pi}\cdot\frac{1}{n^2}(-1)^n - \left(\frac{1}{1+\frac{1}{n^2}}\right)\left(\frac{1}{n^2}\right)$$

$$= \frac{1}{n^2+1}\left(e^{\pi}(-1)^n - 1\right)$$

$$\Rightarrow \quad F.S. = \frac{e^{\pi}-1}{\pi} + \sum_{n=1}^{\infty} \frac{2}{\pi(n^2+1)}\left(e^{\pi}(-1)^n - 1\right)\cos(nx)$$

END OF EXAM